THE STATUS OF RESEARCH IN TURBULENT BOUNDARY LAYERS WITH FLUID INJECTION

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Summary. The purpose of this paper is to discuss the state of research in turbulent boundary layers with fluid injection and to present a starting point for further research in this topic. All available theoretical as well as experimental investigations have been considered. The main assumptions of the theories and the similarities of the various theories to each other are discussed and experimental data are compared with the predictions of the theories.

LIST OF SYMBOLS

\[ A \quad \text{Eq. (2.32) constant} \]
\[ A \quad \text{Eq. (3.18) constant} \]
\[ A \quad \text{Eq. (2.41) integration constant} \]
\[ B \quad \text{Eq. (2.12.1) injection parameter} \]
\[ B \quad \text{Eq. (2.41) integration constant} \]
\[ B \quad \text{Eq. (3.18) integration constant} \]
\[ B_s \quad \text{Section 2.1.2 blowing rate parameter} \]
\[ b \quad \text{Eq. (2.36) Rannie's constant} \]
\[ C \quad \text{Eq. (2.42) constant in law of the wall} \]
\[ c_f = \frac{\tau_w}{\rho_{\infty} u_{\infty}^2} \quad \text{skin friction coefficient} \]
\[ c_F = \frac{1}{x} \int_0^x c_F \, dx \quad \text{total skin friction coefficient} \]
\[ C_F = \frac{\rho_F}{\rho} \quad \text{mass fraction of injected gas} \]
\[ c_h = \frac{q_w}{\rho_{\infty} u_{\infty} c_{p_{\infty}} (T_r - T_w)} \quad \text{Stanton number (compressible flow)} \]
\[ c_h = \frac{q_w}{\rho_{\infty} u_{\infty} c_{p_{\infty}} (T_w - T_{\infty})} \quad \text{Stanton number (incompressible flow)} \]
\[ c_p \quad \text{specific heat at constant pressure} \]
\[ D \quad \text{diffusion coefficient} \]

† Now at Messer Griesheim GmbH, Frankfurt am Main, Germany.
\begin{align*}
D & \quad \text{diameter} \\
d & \quad \text{Eq. (2.44) } \quad \text{integration constant} \\
E & \quad \text{Eq. (3.23) } \quad \text{integration constant} \\
F & = \frac{\varepsilon_w v_w}{\varepsilon_w u_\infty} \\
F & \quad \text{Eq. (3.23) } \quad \text{integration constant} \\
F_C & \quad \text{Eq. (2.15) } \quad \text{similarity parameter} \\
F_{r_x} & \quad \text{Eq. (2.15) } \quad \text{similarity parameter} \\
F_{r_\theta} & \quad \text{Eq. (2.15) } \quad \text{similarity parameter} \\
H & = h + u^2/2 \\
H & = \delta^*/\theta \\
h & \quad \text{boundary layer shape parameter} \\
M & = u/\theta \\
M & \quad \text{Mach number} \\
n_t & \quad \text{Eq. (2.44.2) } \quad \text{parameter of bilogarithmic law} \\
p & \quad \text{static enthalpy} \\
p_t & \quad \text{Eq. (2.44.2) } \quad \text{parameter of bilogarithmic law} \\
Pr & = \frac{\mu c_p}{\lambda} \\
Pr_T & = \frac{\varepsilon_M c_p}{\varepsilon_H} \\
q & = \lambda (\partial T/\partial y) \\
Q & \quad \text{Eq. (3.22) } \quad \text{total heat flux} \\
R & = u_\infty Q_{\infty}/\mu_\infty \\
R & \quad \text{Reynolds number per unit length} \\
R_D & = u_\infty Q_{\infty} D/\mu_\infty \\
R_s & = u_s Q_{\infty}/\mu_s \\
R_x & = u_\infty Q_{\infty} x/\mu_\infty \\
R_\theta & = u_\infty Q_{\infty} \theta/\mu_\infty \\
R_{\delta_s} & = u_\infty Q_{\infty} \delta_s/\mu_\infty \\
r & \quad \text{recovery factor} \\
Sc & = \frac{\mu}{\rho D} \\
Sc_T & = \frac{\varepsilon_M}{\rho \varepsilon_D} \\
s & \quad \text{Reynolds analogy factor} \\
T & \quad \text{absolute temperature} \\
u & \quad \text{velocity in } x\text{-direction} \\
u_x & \quad \text{friction velocity} \\
u_x & \quad \text{Eq. (2.51) } \quad \text{McQuaid's universal velocity} \\
V & \quad \text{Eq. (3.20) } \quad \text{temperature distribution parameter} \\
v & \quad \text{velocity in } y\text{-direction} \\
W & \quad \text{Eq. (3.21) } \quad \text{temperature distribution parameter} \\
x & \quad \text{coordinate along the wall} \\
y & \quad \text{coordinate normal to the wall} \\
y_t & \quad \text{Eq. (2.44.2) } \quad \text{parameter of bilogarithmic law}
\end{align*}
### Greek symbols

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Equation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_i$</td>
<td>Eq. (3.16)</td>
<td>constant</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>Eq. (2.55)</td>
<td>intermittency factor</td>
</tr>
<tr>
<td>$\gamma = c_p/c_v$</td>
<td></td>
<td>ratio of specific heats</td>
</tr>
<tr>
<td>$I'$</td>
<td></td>
<td>molecular weight</td>
</tr>
<tr>
<td>$\Delta^*$</td>
<td>Eq. (2.35)</td>
<td>effective displacement thickness</td>
</tr>
<tr>
<td>$\Delta_l$</td>
<td>Eq. (2.61)</td>
<td>pressure parameter</td>
</tr>
<tr>
<td>$\delta$</td>
<td></td>
<td>boundary layer thickness</td>
</tr>
<tr>
<td>$\delta^*$</td>
<td>$\int_0^\delta (1 - \varrho u/\varrho_\infty u_\infty) , dy$</td>
<td>displacement thickness</td>
</tr>
<tr>
<td>$\delta_0$</td>
<td></td>
<td>value of &quot;boundary layer thickness&quot; at which $u_\infty - u/ u_\tau = 1$</td>
</tr>
<tr>
<td>$\delta_s$</td>
<td></td>
<td>value of &quot;boundary layer thickness&quot; for twice the distance from the wall to the position at which $\gamma = 0.5$</td>
</tr>
<tr>
<td>$\varepsilon_D$</td>
<td>$-\nu' C' / \partial C_F/ \partial y$</td>
<td>eddy diffusion coefficient</td>
</tr>
<tr>
<td>$\varepsilon_H$</td>
<td>$-\varrho c_p T' / \partial T/ \partial y$</td>
<td>eddy conductivity of heat</td>
</tr>
<tr>
<td>$\varepsilon_M$</td>
<td>$-\rho u'v'/ \partial u/ \partial y$</td>
<td>eddy viscosity</td>
</tr>
<tr>
<td>$\eta$</td>
<td>Eq. (2.27)</td>
<td>transformation parameter</td>
</tr>
<tr>
<td>$\theta$</td>
<td>$\int_0^\delta \frac{\varrho u}{\varrho_\infty u_\infty} \left(1 - \frac{u}{u_\infty}\right) , dy$</td>
<td>momentum thickness</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>Eq. (2.42)</td>
<td>mixing length constant</td>
</tr>
<tr>
<td>$\lambda$</td>
<td></td>
<td>thermal conductivity</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>Eq. (2.45)</td>
<td>parameter of bilogarithmic law</td>
</tr>
<tr>
<td>$\lambda_l$</td>
<td>Eq. (3.23)</td>
<td>constants</td>
</tr>
<tr>
<td>$\mu$</td>
<td></td>
<td>viscosity</td>
</tr>
<tr>
<td>$\nu$</td>
<td></td>
<td>kinematic viscosity</td>
</tr>
<tr>
<td>$\xi$</td>
<td>Eq. (2.28)</td>
<td>transformation parameter</td>
</tr>
<tr>
<td>$\theta$</td>
<td></td>
<td>density</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>Eq. (2.26)</td>
<td>transformation parameter</td>
</tr>
<tr>
<td>$\tau$</td>
<td></td>
<td>shear stress</td>
</tr>
<tr>
<td>$\Phi$</td>
<td>Eq. (2.53)</td>
<td>function representing wall effects</td>
</tr>
</tbody>
</table>
1. INTRODUCTION

The possible applications of the compressible turbulent boundary layer with fluid injection cover a wide engineering field; in addition fluid injection has considerable theoretical interest. Transpiration cooling, the fluid injection of a secondary gas through a porous surface into the main stream, results in a deformation of the boundary layer such that skin
friction and heat transfer are reduced. Moreover, the high-speed re-entry of rockets and satellites into the atmosphere requires the development of techniques for protecting their payloads against the intense heat produced. There are three promising ways in which this cooling may take place:

(i) Ablation of heat shield materials ("ablation cooling", see ref. 2).
(ii) Injection of secondary fluid through one or more slots into the main stream ("film cooling", see refs. 3 to 7).
(iii) Injection of secondary gas through a porous surface into the main stream ("transpiration cooling", the subject of the present review).

Transpiration cooling is very similar to the first two processes and a detailed knowledge of the mechanism should therefore assist the understanding of both ablation and film cooling. In the former case, the injection air is replaced by the gaseous decomposition products of the ablating materials (such as glass-reinforced Teflon) and in the latter case by the vapour from the liquid (for example, liquid helium) which is on the surface. Transpiration cooling is a suitable method of cooling any surface exposed to a high-temperature gas stream and typical processes occur in electric arcs, gas turbine blades, combustion chamber walls, hypersonic ram-jet intakes, and in rocket motor nozzles. In all that follows the consideration is restricted to fluid injection.

In the last few years many scientists have turned their attention to the transpired turbulent boundary layer, and a large number of theoretical and experimental analyses have been published. Eckert and Livingood, Sellers, Mager and Divoky have compared transpiration and film cooling with conventional cooling methods like convection cooling. Their investigations showed that cooling methods which influence the temperature and velocity profiles directly, like transpiration cooling, are much more effective than conventional wall cooling. Sometimes it is only necessary to protect certain areas exposed to a supersonic, or very hot air stream. One example is the stagnation point of rockets and satellites (re-
entering the atmosphere) where it is necessary to reduce the wall temperature considerably in those regions where the velocity decreases to zero. This can be effected by constructing the region around the stagnation point from a porous material and forcing air from inside into the main stream. This stagnation point injection would mainly result in a laminar and transitional boundary layer problem and only to a minor extent (in the region further downstream) in a turbulent one. This region further downstream is not necessarily porous so that the turbulent boundary layer developing along this impermeable surface would be a very special case, namely a turbulent boundary layer along a solid wall with a "transpired history" in the laminar and transitional region. A brief survey of this topic is given by Eckert et al.\(^{(4)}\). The downstream effects of transpiration cooling are the subject of papers by Bernicker\(^{(17)}\) and Woodruff and Lorenz\(^{(108)}\).

Craven\(^{(18)}\), Squire\(^{(19)}\) and Spalding et al.\(^{(20)}\) presented reviews of analytical and experimental analyses of the turbulent boundary layer with air and foreign gas injection. Craven simply offers, without comment, a review of all the available theories for determining the skin friction coefficient for boundary layers with suction and injection, whereas Squire presents some calculations to show the advantages of transpiration cooling for aircraft and rockets in the Mach number range of \(2.5 \leq M \leq 5\). Spalding et al.\(^{(20)}\) give a brief systematical survey of the theoretical approaches which predict the skin friction coefficient and the Stanton number variation of the compressible turbulent boundary layer with air and foreign gas injection. But all the information summarized in refs. 18 to 20 must be regarded as rather incomplete since recent investigations are not considered. All the available analytical and experimental information concerning turbulent boundary layers with fluid injection will be summarized in the next sections. The main aim is to establish a survey as up-to-date, and as systematic, as possible, so that further investigations in this topic can begin on the basis of this summary.

2. TURBULENT BOUNDARY LAYERS WITH AIR INJECTION

2.1. Compressible flow

2.1.1 Experimental investigations

The compressible turbulent boundary layer with fluid injection was the subject of quite a number of experimental investigations in the past few years. The most interesting facts about these experimental studies, like geometry of the experimental set-up, injected gas, injection parameter
\[ F = \frac{\dot{m}_w v_w}{\rho_\infty U_\infty}, \] temperature, Mach number, Reynolds number and measured quantities are summarized in Table 1. For the sake of completeness Table 1 includes some information about investigations of minor interest in the present context, like preliminary investigations, and those investigating the influence of injection on heat transfer in supersonic nozzles and rocket motors.

The boundary layer along porous cones or hemispheres were the subject of most of the experimental investigations. The reason for this choice is the possible application of fluid injection in the region of the stagnation point of rockets and satellites, and the relatively easy and fairly accurate determination of overall skin friction coefficients and Stanton numbers from momentum and energy balances for the whole cone. Such a relatively simple arrangement has the disadvantage that valuable information about the local behaviour of the boundary layer cannot be obtained. However, only detailed information about local skin friction and heat transfer coefficients, boundary layer profiles and recovery factors can be considered as reasonable checks for the reliability of the predictions of theories. Moreover, most of the theories are derived for two-dimensional flows along flat plates so that this experimental set-up is more important than the axisymmetrical flow along cones.

Only a few measurements by Rubesin\(^{21}\), Danberg\(^{22}\) and Jeromin\(^{23}\) deal with distributed air injection along a flat plate. Rubesin\(^{21}\) has measured local skin friction coefficients, local Stanton numbers and the influence of injection on the recovery factor for the Mach numbers 0, 2.0 and 2.7. At a much higher Mach number of 6.2 an experimental investigation was undertaken by Danberg\(^{22}\) covering a large range of heat transfer rates at the wall. He evaluated local boundary layer characteristics like skin friction and heat transfer coefficients, momentum and displacement thickness, boundary layer shape parameter \(H\) and the Reynolds analogy factor from measured velocity and temperature profiles. The skin friction and heat transfer coefficients were determined from the slope at the wall of the boundary layer profiles so that \(c_f\) and \(c_h\) show scatter in the order of ±20 per cent. Jeromin's\(^{23}\) investigation at Mach numbers of 2.5 and 3.5 concentrates more on the case of zero and low heat transfer rates at the wall. From measured boundary layer profiles the usual boundary layer parameters \(c_f, \theta, \delta, \delta^*\) and \(H\) have been determined. The skin friction coefficient was evaluated here from the law of the wall for transpired boundary layers in connection with a boundary layer transformation

\(^{\dagger}\) A Mach number of 0 refers to incompressible flow.
### Table 1.
**Experimental Studies of the Compressible Turbulent Boundary Layer with Fluid Injection**

<table>
<thead>
<tr>
<th>Investigator(s)</th>
<th>Geometry</th>
<th>Injected gas</th>
<th>$F \times 10^3$</th>
<th>Temp. $[°K]$</th>
<th>Measured quantity</th>
<th>Mach No.</th>
<th>Reynolds number</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sullivan, Chauvin and Rumsey&lt;sup&gt;(24)&lt;/sup&gt; 1953</td>
<td>8° cone</td>
<td>air</td>
<td>0 to 2</td>
<td>480 ≤ $T_w$ ≤ 535</td>
<td>reduction of wall temperature due to injection</td>
<td>2.05</td>
<td>$R = 8 \times 10^6$ [per metre]</td>
<td>preliminary investigation, not too accurate.</td>
</tr>
<tr>
<td>Rubesin, Pappas and Okuno&lt;sup&gt;(23)&lt;/sup&gt; 1955</td>
<td>flat plate</td>
<td>air</td>
<td>0 to 3</td>
<td>300 ≤ $T_\infty$ ≤ 345</td>
<td>Stanton number, skin friction coefficient, recovery factor</td>
<td>2.7</td>
<td>$1.5 ≤ R_\infty \times 10^{-6}$ ≤ 7</td>
<td>good agreement with Rubesin’s theory</td>
</tr>
<tr>
<td>Chauvin and Carter&lt;sup&gt;(20)&lt;/sup&gt; 1955</td>
<td>8° cone</td>
<td>nitrogen helium</td>
<td>0 to 14, 0 to 0.5, 0 to 0.5</td>
<td>477 ≤ $T_w$ ≤ 533, $T_w/T_\infty$ = 1.5</td>
<td>Stanton number, recovery factor</td>
<td>2.05</td>
<td>$R = 8 \times 10^6$ [per metre]</td>
<td>preliminary investigation, not too accurate</td>
</tr>
<tr>
<td>Rubesin&lt;sup&gt;(21)&lt;/sup&gt; 1956</td>
<td>flat plate</td>
<td>air</td>
<td>0 to 3</td>
<td>300 ≤ $T_\infty$ ≤ 345</td>
<td>Stanton number, local skin friction coefficient, recovery factor, average skin friction coefficient, Stanton number</td>
<td>0</td>
<td>2.0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>cone</td>
<td>air</td>
<td>0 to 3</td>
<td>300 ≤ $T_\infty$ ≤ 345</td>
<td>Stanton number, recovery factor</td>
<td>2.7</td>
<td>2.0</td>
<td></td>
</tr>
<tr>
<td>Leadon and Scott&lt;sup&gt;(24, 25)&lt;/sup&gt; 1956</td>
<td>flat plate</td>
<td>air, helium</td>
<td>0 to 4, 0 to 15</td>
<td>$T_\infty$ = 280</td>
<td>Stanton number, recovery factor</td>
<td>3</td>
<td>$R = 4 \times 10^4$ [per metre]</td>
<td></td>
</tr>
<tr>
<td>Tendeland and Okuno (1956)</td>
<td>good agreement with Rubesin's theory</td>
<td>Stanton number, recovery factor</td>
<td>Stanton number, recovery factor</td>
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<tr>
<td>2.7</td>
<td>4.8</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.8 \times 10^2 \leq R \leq 2.8 \times 10^3 \text{ (per metre)}</td>
<td>\text{297} \leq T_0 \leq 2.4</td>
<td>\text{heat transfer, wall temperature reduction}</td>
<td>\text{local skin friction coefficient}</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>20° cone, Hemisphere</td>
<td>air</td>
<td>\text{air}</td>
<td>\text{Teflon}</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Scott, Anderson and Elgin (1959)</td>
<td>laminar, transitional and turbulent flow, blow-off studies</td>
<td>\text{0 to 32}</td>
<td>\text{0 to 20}</td>
<td></td>
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</table>

<table>
<thead>
<tr>
<th>Leadon, Scott and Anderson (1957)</th>
<th>laminar flow and transition region studied</th>
<th>laminar flow and transition region studied</th>
<th>laminar flow and transition region studied</th>
</tr>
</thead>
<tbody>
<tr>
<td>30° to 70°</td>
<td>0 to 20</td>
<td>0 to 20</td>
<td>0 to 20</td>
</tr>
<tr>
<td>\text{40° double wedge}</td>
<td>7.5</td>
<td>\text{air}</td>
<td>\text{air}</td>
</tr>
<tr>
<td>\text{nitrogen}</td>
<td>\text{helium}</td>
<td>\text{helium}</td>
<td>\text{Teflon}</td>
</tr>
<tr>
<td>Ward and Harmon (1959)</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>Scott and Elgin (1959)</td>
<td>\text{heat transfer, Stanton number, velocity and concentration profiles, recovery factor}</td>
<td>\text{heat transfer, Stanton number, velocity and concentration profiles, recovery factor}</td>
<td>\text{heat transfer, Stanton number, velocity and concentration profiles, recovery factor}</td>
</tr>
<tr>
<td>2×10^4 \leq R \leq 4×10^4 \text{ (per metre)}</td>
<td>\text{297} \leq T_0 \leq 2.4</td>
<td>\text{heat transfer, wall temperature reduction}</td>
<td>\text{local skin friction coefficient}</td>
</tr>
<tr>
<td>\text{flat plate, cone}</td>
<td>\text{helium}</td>
<td>\text{helium}</td>
<td>\text{Teflon}</td>
</tr>
<tr>
<td>Investigator(s)</td>
<td>Geometry</td>
<td>Injected gas</td>
<td>$F \cdot 10^{8}$</td>
</tr>
<tr>
<td>-------------------------------------</td>
<td>--------------------</td>
<td>--------------------</td>
<td>------------------</td>
</tr>
<tr>
<td>Green and Nall$^{(18)}$</td>
<td>supersonic nozzle</td>
<td>air</td>
<td>0 to 12</td>
</tr>
<tr>
<td>Pappas and Okuno$^{(19)}$</td>
<td>$15^\circ$ cone</td>
<td>air/helium/Freon-12</td>
<td>0 to 8/0 to 4/0 to 5</td>
</tr>
<tr>
<td>Bartle and Leadon$^{(27)}$</td>
<td>flat plate</td>
<td>nitrogen</td>
<td>0 to 1.9</td>
</tr>
<tr>
<td>Warner and Emmons$^{(21)}$</td>
<td>cylindrical rocket motor chamber</td>
<td>water, anhydrous ammonia, ethyl alcohol, Freon-113</td>
<td>2600 $\leq T_{\infty} \leq 4100$</td>
</tr>
<tr>
<td>Librizzi and Cresci(^{(13)}) 1963</td>
<td>supersonic nozzle (2-dimensional)</td>
<td>helium nitrogen</td>
<td>895 $\leq T_{\infty} \leq$ 1485</td>
</tr>
<tr>
<td>---</td>
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</tr>
<tr>
<td>Pappas and Okuno(^{(41)}) 1964</td>
<td>15° cone</td>
<td>air, helium, Freon-12</td>
<td>0 to 8, 290 to 417</td>
</tr>
<tr>
<td>McRee, Peterson and Braslaw(^{(42)}) 1964</td>
<td>flat plate</td>
<td>air</td>
<td>0 to 1.5, 300</td>
</tr>
<tr>
<td>Danberg(^{(12)}) 1960, 1964</td>
<td>flat plate</td>
<td>air</td>
<td>0 to 1.8, 300, 4.2 $\leq T_{\infty}/T_{\infty} \leq 5$, 5.5, 4.1 $\leq T_{\infty}/T_{\infty} \leq 5$, 7.6</td>
</tr>
<tr>
<td></td>
<td>flat plate</td>
<td>air</td>
<td>0 to 2.5</td>
</tr>
</tbody>
</table>

*\(T_{\infty}\) is the free-stream temperature, *\(R\) is the recovery factor, *\(R\) is the skin friction coefficient.*
<table>
<thead>
<tr>
<th>Investigator(s)</th>
<th>Geometry</th>
<th>Injected gas</th>
<th>$F \cdot 10^2$</th>
<th>Temp. [°K]</th>
<th>Measured quantity</th>
<th>Mach No.</th>
<th>Reynolds number</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jeromin (^{22}) 1966</td>
<td>flat plate</td>
<td>air</td>
<td>0 to 1.7 to 0 to 2.2</td>
<td>$T_0 = 295$ to $T_w/T_0 \leq 2.3$ to $T_w/T_0 \leq 3.5$</td>
<td>velocity and temperature profiles, local skin friction coefficient, boundary layer parameters</td>
<td>2.5</td>
<td>$1.6 \times 10^7 \leq R_z \leq 2.5 \times 10^7$</td>
<td></td>
</tr>
<tr>
<td>Fogaroli and Saydah (^{43}) 1966</td>
<td>7.5° cone</td>
<td>air</td>
<td>0 to 20 to 1050</td>
<td>$T_0 = 450$ to $T_w/T_0 \leq 3.5$ to $1050$</td>
<td>average skin friction and heat transfer coefficient</td>
<td>5.3 and 8.1</td>
<td>$R = 0.97 \times 10^6$</td>
<td>0.72 x 10^6 [per metre]</td>
</tr>
</tbody>
</table>
RESEARCH IN TURBULENT BOUNDARY LAYERS

(see Section 2.1.5) and checked by the momentum equation so that these values must be considered at present as the most realistic and accurate information about \( c_f \); however, there is still some scatter in the results.

The local reduction of heat transfer due to injection through a porous surface is the subject of quite a number of experimental investigations using the flat plate configuration. These are the experimental studies by Leadon and Scott\(^{(24, 25)}\), Scott, Anderson and Elgin\(^{(40)}\) and Bartle and Leadon\(^{(27)}\) who measured Stanton numbers and recovery factors and the already mentioned investigation by Danberg\(^{(22)}\). All the other investigators considered in Table 1 measured average skin friction coefficients or average Stanton numbers for porous cones. It is rather doubtful whether the measurements of the average skin friction coefficient or Stanton number for a porous cone are a reasonable check for theories predicting the corresponding local quantities for a flat plate even when plots of \( c_f/c_{f0} \) against \( 2F/c_{f0} \), or \( c_f/c_{f0} \) against \( 2F/c_{f0} \), or corresponding plots for \( c_h \) are used which might reduce this uncertainty by assuming that the reduction of the overall skin friction coefficient and Stanton number are the same as for the local quantities under equal conditions.

All available data for the skin friction coefficient and the Stanton number are plotted in Figs. 1 and 2. In Fig. 1 skin friction coefficients for boundary layers with air injection through a porous surface are compared at \( R_x = \) constant with those obtained along solid surfaces by plotting \( c_f/c_{f0} \) against \( 2F/c_{f0} \). Such a simple plot shows best the reduction of the skin friction coefficient due to injection compared with the case of zero injection which is characterized in Fig. 1 by the point \( 2F/c_{f0} = 0 \) and \( c_f/c_{f0} = 1 \). The data for the incompressible flow considered in Fig. 1 will be discussed in Section 2.2.1. Despite the scatter of the points the tendency is clear: the reduction of the skin friction coefficient due to air injection for \( 2F/c_{f0} = \) constant becomes smaller with increasing Mach number.\(^{†}\)

The four regions marked in Fig. 1 represent the reduction of the skin friction coefficient for the Mach numbers \( M = 0, M = 2.5, M = 3.5 \) and \( M = 6.2 \). Only data for skin friction coefficients measured along flat plates have been considered when the borders of the regions were fixed, namely the investigations by McQuaid\(^{(28)}\) (\( M = 0 \)), Rubesin\(^{(21)}\) and

\(^{†}\) It should be noticed that \( c_{f0} \) and \( c_{h0} \) decrease with increasing Mach number as well so that it cannot be directly deduced from plots of \( c_f/c_{f0} \) against \( 2F/c_{f0} \), \( c_h/c_{h0} \) against \( F/c_{h0} \), respectively, that fluid injection becomes less effective with increasing Mach number. Plotting, for example, \( c_f/c_{f0} \) against \( F \) gives very roughly about the same reduction of the skin friction coefficient due to injection independent of the Mach number.
Jeromin\textsuperscript{(29)} ($M = 2.5$, $M = 3.5$) and Danberg\textsuperscript{(22)} ($M = 6.2$). Regions rather than lines have been chosen to represent the experimental data because of their scatter which becomes considerable for Danberg's data at $M = 6.2$. Only Rubesin's\textsuperscript{(21)} and Tendeland and Okuno's\textsuperscript{(30)} data for porous cones fit with some good will in these four regions. The measurements of Pappas and Okuno\textsuperscript{(31)} at a porous cone show a reasonable agreement with other experimental investigations only for $M = 0.3$ and $M = 0.7$, whereas their data for $M = 3.5$ and $M = 4.5$ give a much smaller reduction of the skin friction coefficient than found by other investigators.

It must be stressed in this connection that two parameters cannot be considered in all details in Fig. 1: the Reynolds number $R_x$ and the temper-
ature ratio $T_w/T_\infty$. Regarding their influence one must expect regions of reduction of the skin friction coefficient in Fig. 1 rather than lines for $M = \text{constant}$. But the scatter of the published data does not allow an estimation of the influence of $R_x$ or $T_w/T_\infty$ on the reduction of $c_f$ in such a plot apart from the fact that the range for $R_x$ and $T_w/T_\infty$ so far covered by the experimental investigations is far too small. There are scarcely any experimental analyses which concentrate systematically on these parameters. Only McQuaid's data at $M = 0$ can be employed to draw the conclusion that the influence of the Reynolds number in a plot $c_f/c_{fo} = f(2F/c_{fo})$ seems to be negligible. But nevertheless Fig. 1 should give first of all a rough idea about the reduction of the skin friction coefficient due to injection, so that the choice of these four regions might be justified

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Investigator</th>
<th>Mach number</th>
<th>Geometry</th>
</tr>
</thead>
<tbody>
<tr>
<td>+</td>
<td>Mickley and Davies (^{(44)}) 1957</td>
<td>incompressible</td>
<td>flat plate</td>
</tr>
<tr>
<td>×</td>
<td>Romanenko and Kharchenko (^{(16)}) 1963</td>
<td>incompressible</td>
<td>flat plate</td>
</tr>
<tr>
<td>*</td>
<td>McQuaid (^{(68)}) 1966</td>
<td>incompressible</td>
<td>flat plate</td>
</tr>
<tr>
<td>▽</td>
<td>Rubesin (^{(33)}) 1956</td>
<td>0; 2·0; 2·7</td>
<td>flat plate</td>
</tr>
<tr>
<td>□</td>
<td>Danberg (^{(32)}) 1960</td>
<td>5·1</td>
<td>flat plate</td>
</tr>
<tr>
<td>■</td>
<td>Danberg 1964</td>
<td>6·2</td>
<td>flat plate</td>
</tr>
<tr>
<td>○</td>
<td>Jeromin (^{(32, 48)}) 1966</td>
<td>2·5; 3·5</td>
<td>flat plate</td>
</tr>
<tr>
<td>△△△△△</td>
<td>Pappas and Okuno (^{(31)}) 1960</td>
<td>0·3; 0·7; 3·5; 4·5</td>
<td>cone</td>
</tr>
<tr>
<td>△</td>
<td>Rubesin (^{(33)}) 1956</td>
<td>0; 2·0; 2·7</td>
<td>cone</td>
</tr>
<tr>
<td>▲</td>
<td>Tendeland and Okuno (^{(30)}) 1956</td>
<td>2·7</td>
<td>cone</td>
</tr>
</tbody>
</table>
to represent the experimental data in the next sections where they are needed for comparisons with available theories.

In Fig. 2 the available experimental data about the reduction of the Stanton number due to air injection have been summarized. Despite the scatter the data show corresponding to Fig. 1 a clear tendency: the reduction of the Stanton number due to air injection for $F/c_{\infty} = \text{constant}$ becomes smaller with increasing Mach number.† Four regions for $M = 0$, $M = 2$, $M = 3$ and $M = 6$ have been outlined in Fig. 2. They are based on experimental data for the flat plate configuration taking again into consideration the scatter of the published data. These four regions will

† See footnote on page 77.
EXPLANATION OF THE SYMBOLS USED IN FIG. 2

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Investigator</th>
<th>Mach number</th>
<th>Geometry</th>
</tr>
</thead>
<tbody>
<tr>
<td>✗</td>
<td>Romanenko and Kharchenko((33)) 1963</td>
<td>incompressible</td>
<td>flat plate</td>
</tr>
<tr>
<td>+</td>
<td>Leadon and Scott((24, 25)) 1956</td>
<td>3.0</td>
<td>flat plate</td>
</tr>
<tr>
<td>□</td>
<td>Rubesin, Pappas and Okuno((33)) 1955</td>
<td>2.7</td>
<td>flat plate</td>
</tr>
<tr>
<td>○</td>
<td>Bartle and Leaden((37)) 1960</td>
<td>2.0, 3.2</td>
<td>flat plate</td>
</tr>
<tr>
<td>△</td>
<td>Pappas and Okuno((33)) 1964</td>
<td>0.7, 3.67, 4.35</td>
<td>cone</td>
</tr>
<tr>
<td>◄</td>
<td>Scott, Anderson and Elgin((40)) 1959</td>
<td>3.0, 5.0</td>
<td>flat plate, cone</td>
</tr>
<tr>
<td>■</td>
<td>Danberg((22)) 1964</td>
<td>6.2</td>
<td>flat plate</td>
</tr>
</tbody>
</table>

represent the experimental data in the next sections when the predictions of the theories for the reduction of the Stanton number due to injection will be discussed. The experimental data published by Pappas and Okuno\((32)\) and by Scott, Anderson and Elgin\((40)\) agree reasonably well with those measured along flat plates and lie certainly within the scatter of these data.

The scatter for the Stanton number in Fig. 2 is smaller than for the skin friction coefficient in Fig. 1. The reason might be the surface roughness of the porous plates, which differs considerably from one investigator to another, or the fact that different wind tunnels were used by the various investigators. Each investigator used different porous materials for his experiments. It is well known that the skin friction coefficient is influenced to a much greater extent by the surface roughness than the Stanton number.

The influence of air injection on the recovery factor is shown in Fig. 3 where \(r/r_0\) (\(r_0\) is assumed to be 0.89) is plotted against the injection parameter \(F\). Bartle and Leadon’s data suggest a slight reduction of the recovery
factor due to injection whereby the reduction decreases with increasing Mach number for \( F = \text{constant} \). Despite the scatter of the data this tendency is clear. This result is confirmed by the experimental data published by Rubesin, Pappas and Okuno\(^{33}\), Leadon and Scott\(^{24, 25}\) and Pappas and Okuno\(^{31}\). Moreover, Rubesin, Pappas and Okuno\(^{33}\) found an additional influence of the Reynolds number on the reduction of the recovery factor such that the recovery factor drops slightly with increasing Reynolds number for \( F = \text{constant} \). This result can be con-

![Figure 3](image)

**Fig. 3.** The reduction of the recovery factor due to air injection (experimental data).

firmed to a certain extent by measurements of the recovery factor along a solid flat plate (see for example Stalder, Rubesin and Tendeland [136] or Schubauer and Tchen [137]). These experimental results have shown that the recovery factor for a Mach number of \( M_\infty = 2.4 \) drops slightly from \( r = 0.90 \) at \( R_x = 1 \times 10^8 \) to \( r = 0.885 \) at \( R_x = 6 \times 10^8 \). As the Reynolds number range for the experiments considered is about the same for the solid flat plate case and the porous plate one (see Fig. 3 and Table I) it can be deduced that the Reynolds number effect on the recovery factor is less pronounced but still existing for boundary layers along solid surfaces. It must be stressed considering the present experience that it is mainly speculation rather than sound knowledge to pro-
vide a proper interpretation for the increased reduction of the recovery factor for transpired boundary layers; hence an acceptable physical explanation has not been formulated yet. Apart from unavoidable experimental difficulties and possibly errors or at least uncertainties in determining the recovery factor for transpired boundary layers—the scatter of the data for the recovery factor is much wider for the case of boundary layers along permeable surfaces compared with those along solid ones—the most likely reason might be the surface roughness of the porous plates which is several orders higher than for ground surfaces normally used for boundary layer experiments in supersonic wind tunnels and which might produce a different energy spectrum close to the wall.

2.1.2. The reference state concept

The simplest approach for predicting the variation of the boundary layer parameters with Reynolds number, for example, is the reference state concept. Two examples of the compressible turbulent boundary layer with fluid injection are the methods proposed by Nash(44) and Knuth and Dershin(45).

Nash introduced a simple empirical similarity parameter with which he succeeded in correlating compressible experimental data with Turcotte’s sublayer hypothesis for incompressible flow. He modified Turcotte’s injection parameter \( \frac{v_w^*}{u_{r0}} \) into

\[
\frac{\theta_w^2 v_w}{\theta_m u_{r0}} = \frac{u_{\infty}}{u_{r0}} \cdot F
\]

assuming that the new parameter suppresses the effect of compressibility. Plotting \( \frac{\tau_w}{\tau_{w0}} \) against \( \frac{u_{\infty}}{u_{r0}} \cdot F \) he found that the experimental data of Tendeland and Okuno(30) and Pappas and Okuno(32) for compressible flow and the data of Mickley and Davies(44) for incompressible flow lie within a scatter on one single curve, namely

\[
\frac{\tau_w}{\tau_{w0}} = \exp \left[ -6.94 \left( \frac{u_{\infty}}{u_{r0}} \cdot F \right) \left( 1 + \sqrt{\frac{\tau_{w0}}{\tau_w}} \right) \right]
\]

But it must be stressed that this correlation could be fortuitous, moreover Turcotte’s approach is suspect for reasons which will be discussed in the next section. In Nash’s concept there are too many questionable simplifications without any theoretical basis: for example

(i) In Turcotte’s theory, the skin friction velocity must be calculated from the local skin friction coefficient, whereas Nash used for his

---

\( ^* \) Turcotte’s sublayer hypothesis will be discussed in more detail in Section 2.2.
correlation the overall skin friction coefficient as measured for a porous cone in a supersonic flow.

(ii) Experimental data measured on a cone are compared with a theory derived for the case of a flat plate.

(iii) The compressibility effects are only considered by a slight modification of one coordinate, namely by changing the incompressible variable \( \frac{v^*}{u_{r0}} \) into \( \frac{u_\infty}{u_{r0}} \cdot F \).

Knuth and Dershin\(^{(45)}\) tried to suppress the compressibility effects in a more sophisticated way by extending Eckert's\(^{(47)}\) and Sommer and Short's\(^{(48)}\) reference state concept to compressible transpired boundary layers. The usual simplifications of the boundary layer model in the presence of fluid injection (see Section 2.1.3) led Knuth and Dershin to the conclusion that it might be possible to correlate compressible and incompressible skin friction coefficients by plotting \( \frac{c_f^*}{c_{f0}^*} \) against

\[
B^*_f = 11.5F \left( \frac{2}{c_f^*} \cdot \frac{2}{c_{f0}^*} \right)^{1/4}
\]

where the superscript * refers to reference condition. If incompressible and compressible data collapse in such a plot on one single curve—the so-called constant property curve—the reference concept might be justified.

Knuth and Dershin determined their constant property curve from incompressible data (Goodwin\(^{(51)}\) and Smith\(^{(52)}\)). The compressible flow is represented only by the heat transfer data of Bartle and Leadon\(^{(27)}\) since it was believed that heat transfer measurements in supersonic flow are much more reliable than skin friction measurements. This advantage of greater accuracy of the heat transfer data is at least lost by the fact that they had to introduce a modified Reynolds analogy with all its uncertainties in order to relate heat transfer with skin friction coefficients. The reference state expression for turbulent flow with air injection necessary to reduce compressible into incompressible flow is the same as that developed by Knuth\(^{(49)}\) for laminar flow, namely

\[
T^* = 0.5(T_w + T_\infty) + 0.2(T_r0 - T_\infty) + 0.1 \frac{\partial_p \nu_w}{\rho p c_{f0}^*} \frac{1}{c_{f0}^*} (T_w - T_\infty)
\]

This assumption is based on experience obtained for compressible flow without injection where the reference temperature formulae for laminar flow correlate more or less successfully the quantities for turbulent flow as well.
Introducing their Reynolds analogy and applying the reference temperature formulae Knuth and Dershin succeeded in correlating the compressible and the incompressible data on one single line as shown in Fig. 4 where \( \frac{c_f}{c_{f0}} \) is plotted against \( B'_s \). This result seems to justify the reference state concept. A similar good agreement was obtained for the Stanton number whereas the prediction of the reference state concept for the recovery factor is not convincing since its predicted reduction is much bigger than found by experimental investigations.

Finally one can say that much more extensive studies in this field are necessary before final conclusions can be drawn about the reference temperature concept in its present form. The two key questions are

(i) is the reference temperature formula for laminar flow generally valid for turbulent boundary layers? and  
(ii) is there a Reynolds analogy between heat and momentum transfer?

These questions can be solved at present only by appropriate experiments.

Anyway the reference concept must be always considered as a pseudo-theoretical approach which certainly becomes obsolete when a general theory for turbulent boundary layers with air injection has been formulated.
2.1.3. *Theories based on mixing length concept*

A number of theories have been proposed based on the assumptions that the turbulent boundary layer can be divided into a laminar sublayer and an adjoining fully turbulent region. The first approaches in this direction were the boundary layer analyses presented by Rannie\(^{53}\) and Crocco\(^{54}\). They assumed a Couette flow model (all changes in \(x\)-direction are negligible compared with changes in \(y\)-direction) and a simplified Reynolds analogy between momentum and energy transfer. Both theories are based on a simple film model, replacing the boundary layer by a film of constant thickness, and hence neglecting any variations in flow direction as compared with those across the film. The aim of both theories is to establish relations with which the heat transfer coefficient (and hence the necessary amount of cooling medium to protect the surface) can be calculated from the skin friction coefficient which is assumed as known. Despite the fact that the skin friction coefficient was evaluated from isothermal, incompressible turbulent boundary layer formulae, the theoretical predictions agree reasonably well with experimental results.

More sophisticated contributions to a theoretical analysis of the compressible turbulent boundary layer with air injection are the papers by Rubesin\(^{55}\) and Dorrance and Dore\(^{56}\). Considering the present state of knowledge of transpired turbulent boundary layers their approach of solving the boundary layer equations with Prandtl's mixing length theory must be considered as the best possible theoretical solution in integrating directly the boundary layer differential equations. Both theories have much in common in that they are practically all extensions of earlier compressible boundary layer theories for turbulent flow without injection where the latter ones are extended incompressible mixing-length theories taking into account compressibility effects. The starting point for both theories are the boundary layer equations, namely the

\[
\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0
\]  

(2.4)

momentum equation

\[
\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{dp}{dx} + \frac{\partial \tau}{\partial y}
\]  

(2.5)

with

\[\tau = (\mu + \epsilon_M) \frac{\partial u}{\partial y}, \quad \epsilon_M = -\frac{\rho u'v'}{\partial u/\partial y},\]
energy equation
\[ q u \frac{\partial H}{\partial x} + q v \frac{\partial H}{\partial x} = \frac{\partial}{\partial y} (q + ut) \quad (2.6) \]

with
\[ q = \frac{\lambda + \varepsilon_H}{c_p} \frac{\partial h}{\partial y}, \quad \varepsilon_H = -\frac{\varepsilon_p v' T'}{\partial T} \]

These three equations have to be solved simultaneously. The two theories proposed by Rubesin and by Dorrance and Dore are almost identical. Their main assumptions are:

(i) the boundary layer is divided in the usual way into a laminar sublayer close to the wall where molecular transport processes predominate, and a fully turbulent region which is governed by turbulent transport processes only. No buffer layer is considered;

(ii) the turbulent Prandtl number \( Pr_T = \frac{\varepsilon_M c_p}{\varepsilon_H} \) is equal to unity;

(iii) a Couette flow model is assumed, so that the partial differential equations describing the boundary layer behaviour reduce to ordinary ones;

(iv) the gases are ideal;

(v) the shear distribution through the boundary layer can be approximated by Prandtl's mixing length theory, namely

\[ \varepsilon_M = \rho k y^a \left( \frac{\partial u}{\partial y} \right) \]

(vi) only the case of constant pressure \( dp/dx = 0 \) is considered;

(vii) the injection mass flow \( F \) is constant along the surface.

With the help of assumption (iii) the partial differential equations (2.4) and (2.5) can be simplified to an ordinary one, namely

\[ \varepsilon_M v w \frac{d u}{d y} = \frac{d}{d y} \left[ (\mu + \varepsilon_M) \frac{d u}{d y} \right]. \quad (2.7) \]

Setting the Prandtl numbers for the sublayer and the outer region equal to unity leads to a direct relationship between velocity and temperature so that only (2.6) and (2.5) have to be solved simultaneously and eq. (2.6) is not required.

Neglecting the eddy transport term \( \varepsilon_M \) in the sublayer and the viscous term \( \mu \) in the outer region leads to expressions defining the velocity
distribution throughout the boundary layer. It follows for the sublayer
(0 \leq y \leq y_s, subscript s refers to conditions at the edge of the sublayer)
\[ \mu \frac{du}{dy} = \rho \nu w u + \text{constant} = \rho \nu w u + \tau_w \] (2.8)
and for the outer part
\[ \rho \nu w u = \rho \chi^2 y^2 \left( \frac{du}{dy} \right)^2 + \text{constant} \] (2.9)
after introducing Prandtl's mixing length theory. The velocity and shear
stress distribution must be continuous at the edge of the sublayer, hence
the constants in (2.8) and (2.9) must be identical. Therefore one obtains
\[ \rho \nu w u + \tau_w = \rho \chi^2 y^2 \left( \frac{du}{dy} \right)^2 \] (2.10)
or on integrating
\[ y = y_s \int_{u_s}^{u} \frac{\kappa \theta^{1/4} du}{\sqrt{\rho \nu w u + \tau_w}} \text{ for } y \geq y_s. \] (2.11)
A corresponding expression can be derived for the sublayer region. With
the velocity distribution throughout the boundary layer known, the skin
friction coefficient can be determined from the momentum equation
\[ c_F + F = 2 \frac{d\theta}{dx} \] (2.12)
with the momentum thickness
\[ \theta = \int_{\delta}^{\delta} \frac{\rho u}{\rho u_{\infty}} \left( 1 - \frac{u}{u_{\infty}} \right) dy. \]
Up to this point Rubesin's and Dorrance and Dore's theories are
almost identical. The only differences between the theories are the extrap-
opolation of the constants \(\kappa, y_s,\) and \(u_s,\) and the process of the numerical
integration of eq. (2.11) itself. Rubesin made three empirical assumptions
based on measurements in incompressible flow:

(i) the sublayer "Reynolds number" \(u_s y_s = 13.1^2 = \text{constant};\) the
factor \(13.1^2\) is evaluated from experimental results in incompress-
ible flow without injection;
(ii) \( u_s = 13.1 \) for \( T_w = T_\infty \) and \( u_s = 13.1\sqrt{T_w/T_\infty} \) for \( T_w \neq T_\infty \) and \( y_s \) according to eq. (2.11);
(iii) the mixing length constant is not influenced by compressibility effects and fluid injection at the wall and was set \( \kappa = 0.392 \).

Assuming an error of less than 1 per cent Rubesin used only the velocity distribution in the outer part of the boundary layer, i.e. eq. (2.11), to determine the momentum thickness and neglected the contribution of the sublayer. Thus \( u_s \) is made zero and \( y_s \) remains finite in such an integration process. The momentum thickness known, the skin friction coefficient was calculated from the momentum equation (2.12). The result is shown and compared with experimental data in Fig. 5 by plotting \( c_f/c_{f0} \) against \( 2F/c_{f0} \). The theory predicts a reduction of \( c_f \) due to injection but unfor-

![Fig. 5. The reduction of the skin friction coefficient due to air injection (Rubesin's theory in comparison with experimental data).](image-url)
Unfortunately not in the right order of magnitude. The agreement is poor, for both incompressible flow ($M = 0$) and compressible flow ($M = 4, T_w \propto T_r$) especially for high injection rates. The discrepancy is mainly due to the fact that the constants were evaluated from available information about incompressible flow without injection, since there was barely any information about transpired turbulent boundary layers accessible at the time when the theory was derived. But as more recent experiments have shown, these constants are in fact functions of the injection rate and heat transfer. Danberg\textsuperscript{(22)} for example, modified these theories by changing the constants according to his measurements and succeeded in transforming\textsuperscript{†} his measured velocity profile into a form similar to the semilogarithmic profile of incompressible flow, the so-called law of the wall. His coordinates are essentially derived from Rubesin's and Dorrance and Dore's theories, or are at least based on the same boundary layer model. He obtained reasonable agreement between his experimental and theoretical profiles for the fully turbulent region by adjusting his 'constants' of integration of the mixing length velocity profile so as to make them increase with heat transfer and decrease with increasing mass transfer rate at the wall.\textsuperscript{‡}

It must be pointed out that two parameters influence the representation of the theory in Fig. 5, namely the Reynolds number $R_w$ and the temperature ratio $T_w/T_\infty$. Their influence on the skin friction coefficient is shown on Figs. 6 and 7. With increasing Reynolds number for $T_w/T_\infty = \text{const.}$ and $M_\infty = \text{const.}$ Rubesin's theory predicts an increasing reduction of the skin friction coefficient due to injection (see Fig. 6). The influence of heat transfer at the wall is such that with increasing cooling ($T_w \to T_\infty$) the reduction of the skin friction coefficient increases slightly (see Fig. 7). These influences cannot be checked at present by experimental data because the influence of the different parameters on the boundary layer growth have not been investigated systematically enough.

Dorrance and Dore determined the momentum thickness numerically and included the contribution of the sublayer for the velocity distribution in the integration process. The integration constants were evaluated by

---

\textsuperscript{†} Danberg called his theoretical analysis a transformation. His approach is certainly not a transformation in the same sense as used by Coles\textsuperscript{(22)} and Jeromin\textsuperscript{(28)} where the compressible boundary layer equations are transformed mathematically into their corresponding incompressible forms.

\textsuperscript{‡} Another theoretical approach, where the constants are the same as for incompressible flow without injection, has been proposed by Stevenson\textsuperscript{(61)}. His approach is also based on Prandtl's mixing length theory, but the calculation follows a different path so that the final result is different. This theory will be discussed in Section 2.2.3.2.
assuming that the resulting skin friction law eventually derived from eq. (2.12) must reduce to Karman's skin friction law for incompressible flow when $T_w = T_\infty$ and $M_\infty = 0^{(87)}$. The mixing length constant $\alpha$ was assumed again to be unchanged by compressibility effects and fluid injection and was set to 0.393. The main result of Dorrance and Dore's theory for the skin friction coefficient is shown on Fig. 8 and compared with available experimental data. The reduction of the skin friction coefficient due to injection is again predicted by the theory but unfortunately only in the right order of magnitude for the incompressible flow with $M_\infty = 0$ whereas a far bigger reduction of the skin friction coefficient is predicted for compres-
Fig. 7. The influence of the temperature ratio $T_w/T_\infty$ on the reduction of the skin friction coefficient (air injection: Rubesin's theory $M = \text{const.}$, $R_e = \text{const.}$).

Possible flow, represented in Fig. 8 by a Mach number of 5, and heat transfer at the wall ($T_w = T_\infty$) than found by experimental investigations.

Rubesin, and Dorrance and Dore extended their theories to include the case of heat transfer at the wall by assuming a Reynolds analogy with the heat flux proportional to the transport of momentum. In the second part of his analysis Rubesin investigated the influence of fluid injection on the Stanton number now allowing the Prandtl numbers for the sublayer and the outer region to vary throughout the boundary layer. So he determined the relation between the local Stanton number and the local skin friction coefficient when $Pr$ and $Pr_T$ are not unity by taking instead the ratio of corresponding terms of the energy and momentum equation. The influence
of the Prandtl number on the Stanton number is adopted from investigations without injection (see Rubesin(58)) assuming that the skin friction coefficient is independent of the Prandtl number. This process leads to rather complicated ordinary differential expressions for the Stanton number which he solved numerically assuming again that the contribution of the sublayer is negligible compared with the contribution of the outer region. The result for the Stanton number is presented by Rubesin in the form of graphs $c_h = f(R_x)$ with the parameters $M_\infty$, $T_w/T_\infty$ and $F$. Only a typical example for the reduction of the Stanton number due to air injection will be shown here on Fig. 9 where $c_h/c_{h0}$ is plotted against $F/c_{h0}$ for $R_x = 10^7$, $M = 0$ and $M = 4$ and compared with experimental data. A slightly smaller reduction of the Stanton number is predicted by

![Figure 8](image_url)

**Fig. 8.** The reduction of the skin friction coefficient due to air injection (comparison of Dorrance and Dore's(64) theory with experimental data).
Rubesin’s theory for incompressible flow than shown by experimental data. The agreement between theory and experiment is excellent for a Mach number of 4. The curve representing Rubesin’s theory for \( M = 4 \) must be considered as a mean curve for \( 10^6 \leq R_x \leq 10^8 \) and \( T_\infty \approx T_w \approx T_r \). The prediction of Rubesin’s theory for the recovery factor does not agree with experimental data.

Dorrance and Dore extended their theory to include the case of heat transfer at the wall by assuming a complete Reynolds analogy

\[
\frac{c_h}{c_{h0}} = \frac{c_f}{2}
\]

and

\[
r = 1 = \text{constant}.
\]
Their result is shown for $M = 0$ and $M = 5$ with $T_w = T_{\infty}$ and $9 \times 10^5 \ll R_x \ll 3.3 \times 10^6$ in Fig. 10. The agreement is excellent for incompressible flow whereas the prediction for compressible flow is wrong probably due to the fact that the constants were evaluated for incompressible flow without injection. Moreover the choice of a complete Reynolds analogy as expressed by eq. (2.13) becomes rather suspect for compressible flow as several investigations have shown. Instead a Reynolds analogy factor

$$s = \frac{c_f/2}{c_h}$$

should be used which is in most cases smaller than one. Finally one can say that Dorrance and Dore's theory can be recommended only for in-

![Fig. 10. The reduction of the Stanton number due to air injection (comparison of Dorrance and Dore's theory with experimental data).](image-url)
compressible flow whereas Rubesin's theory gives reasonable results only for compressible flow. The experimental data suggest a far bigger influence of the Mach number on the reduction of the skin friction coefficient due to injection than predicted by either theory.

A theory very similar to these of Rubesin and Dorrance and Dore was proposed by Ness\(^{[59]}\). This theory is derived for foreign gas injection, but Walker and Schumann\(^{[60]}\) have shown that it can be used for air injection as well. Ness's analysis is an improvement over Rubesin's since it employs no mathematical approximations in solving the Couette type boundary layer equations. The usual starting condition that the skin friction coefficient and hence the velocity at the interface approach infinity at the leading edge of a plate for example, has been replaced by a more realistic starting condition assuming that the velocity at the interface between the laminar sublayer and the fully turbulent region is equal to the velocity at the edge of the boundary layer at the start of the turbulent region. Despite these improvements, the agreement between theory and experiment, as shown by Walker and Schumann\(^{[60]}\), is still not very convincing (but better than for Rubesin's theory, especially for incompressible flow) perhaps mainly due to the choice of the constants which are assumed again to be independent of the injection rate. The discrepancy between theory and experiment becomes more and more significant as the injection rate and the Mach number increase.

Finally one can draw the conclusion that Rubesin's, Dorrance and Dore and Ness's theories can be improved by considering a buffer layer between the laminar sublayer and the fully turbulent region and adjusting the constants according to the information now available. The 'constants' are most probably functions of the wall temperature and the injection rate, so that this influence must be considered during the integration process. Moreover the Prandtl number could be allowed to vary through the laminar sublayer and the turbulent outer region. But this approach would increase the mathematical problems enormously so that the most interesting application of these possible extended theories would be rather difficult.

### 2.1.4. Analysis based on similarity parameters

The skin friction coefficient for compressible turbulent boundary layers with zero injection and zero pressure gradient is in general a function of the Reynolds number \(R_0\) based on the momentum thickness, the temperature ratio \(T_w/T_\infty\) and the free stream Mach number; hence one can write

\[
c_{f0} = c_{f0}(R_0, T_w/T_\infty, M_\infty).
\]
Moreover most theoretical expressions for the skin friction coefficient can be reduced to the form (see for example Spalding (63))

\[ \frac{c_f \rho}{2} F_c = \psi_\theta(R_\theta \cdot F_{R\theta}) \]  \hspace{1cm} (2.15)

or

\[ \frac{c_f \rho}{2} F_c = \psi_x(R_x \cdot F_{Rx}) \]  \hspace{1cm} (2.15.1)

where the functions \( \psi_\theta \) and \( \psi_x \) are independent of Mach number and temperature ratio. The influence of these two parameters is only considered by the functions \( F_c \) and \( F_{R\theta} \) (or \( F_{Rx} \) respectively) which were eventually found from experimental data and set

\[ F_c = \left[ \sqrt[3]{\int_0^1 \sqrt{\frac{\theta}{\theta_\infty}} \frac{d}{u} \left( \frac{u}{u_\infty} \right)} \right]^{-2} \]  \hspace{1cm} (2.16)

and

\[ F_{R\theta} = \frac{\mu_\infty}{\mu_w} \]  \hspace{1cm} (2.17)

The main advantage of treating the boundary layer in such a way is the resulting relatively simple calculation process for skin friction and heat transfer coefficients. For example, only two diagrams are necessary to determine \( c_f \): one containing \( c_f \cdot F_c \) as a function of \( R_\theta \cdot F_{R\theta} \) or \( R_x \cdot F_{Rx} \) which is universal and hence independent of \( M_\infty \) and \( T_w/T_\infty \) (see Fig. 11); the other containing \( F_c \cdot F_{R\theta} \) and \( F_{Rx} \) as functions of the parameter \( M_\infty \) and \( T_w/T_\infty \). Another advantage is the fact that the \( F_{R\theta} \)- and \( F_{Rx} \)-functions were chosen so as to fit best available experimental data and are not based on more or less arbitrary assumptions.
Spalding et al. extended this skin friction calculation method to transpired turbulent boundary layers, postulating that $F_C$, $F_{R0}$ and $F_{Rx}$ are additionally a function of an injection parameter defined by $B = \frac{\frac{\partial w}{\partial x} v_w u_{\infty}}{\tau_w}$ so that

$$F_C = F_C(M_{\infty}, T_w/T_{\infty}, B)$$

$$F_{R0} = F_{R0}(M_{\infty}, T_w/T_{\infty}, B)$$

$$F_{Rx} = F_{Rx}(M_{\infty}, T_w/T_{\infty}, B).$$

The last two equations are linked by the momentum equation (2.12) which can be written in the form

$$\frac{dR_\theta}{dR_x} = \frac{c_f}{2} (1 + B).$$

(2.12.1)

$F_C$, $F_{R0}$ and $F_{Rx}$ can be regarded as constants when $M_{\infty}$, $T_w/T_{\infty}$ and $F$ are independent of $x$, so that (2.12.1) can be written as

$$\frac{d(R_\theta \cdot F_{R0})}{d(R_x \cdot F_{Rx})} = \frac{c_f}{2} \frac{(1 + B) F_{R0}}{F_{Rx} \cdot F_C}$$

(2.12.2)

which is the wanted connection between $F_{R0}$ and $F_{Rx}$. The problem is now the determination of the functions $F_C$ and $F_{R0}$ or $F_{Rx}$ in terms of $M_{\infty}$, $T_w/T_{\infty}$ and $B$. In principle systematical analyses of experimental data would lead to these functions plotting $c_f$ against $R_x$ or $R_\theta$ for fixed values $M_{\infty}$, $T_w/T_{\infty}$ and $B$. But unfortunately there is not enough information available to permit this procedure. Instead Spalding et al. analysed available theories (Rubesin, Dorrance and Dore, Clarke, Menkes and Libby, Rubesin and Pappas, Turcotte, Lapin and Denison) with their similarity parameter concept and found that in most theories $F_C$ can be expressed by

$$F_C = \left[ \int_0^1 \left( \frac{\partial \theta}{\partial \infty} \right)^{1/2} \left( 1 + B \frac{u}{u_{\infty}} \right) \right]^{-2}. $$

(2.21)

† The parameter $B$ was introduced by Dorrance. With the help of this parameter the shear stress distribution through the boundary layer can be described relatively simply by

$$\frac{\tau}{\tau_w} = 1 + B \frac{u}{u_{\infty}}.$$
Equation (2.21) has been solved in a rather complicated calculation process. The result is presented by Spalding et al. in the form of tables and graphs where $F_C$ is given in terms of $(1 + B)$ for the parameter $M_\infty$ and $T_w/T_\infty$. The ranges of variables covered by their calculation are

\begin{align*}
0 & \leq M_\infty \leq 12 \\
0.05 & \leq T_w/T_\infty \leq 20 \\
0 & \leq B \leq 19.
\end{align*}

It must be pointed out here that the extension of the present calculation process to hypersonic flow with $M_\infty \sim > 5$ is rather questionable since real gas effects are not considered. Analysing the available theories to obtain an expression for $F_{R\theta}$ was not successful since the resulting relationships

\[ F_{R\theta} = \frac{\mu}{P_{\infty}} (1 + B)^{-\gamma/2} \]

\[ \text{Note: These data for helium to air injection: all others for air to air.} \]

Fig. 12. The determination of the $F_{R\theta}$ function after Spalding et al. (20)

† See also Section 3.4.
Fig. 13. Comparison of experimental data with the universal turbulent boundary layer law after Spalding et al.\(^\text{680}\)
Fig. 14. The reduction of the skin friction coefficient due to air injection (comparison of Spalding et al.'s theory with experimental data.)

differ considerably from theory to theory so that only the analysis of experimental data remains. Assuming the equations (2.18) to (2.20) hold, the universal plot on Fig. 11 can be used to analyse the available experimental data and to determine $F_{R\theta}$ as a function of $B$. The result is shown on Fig. 12. There is obviously a Mach number effect in Fig. 12 with $M_\infty$ increasing in the direction of the arrow when one only considers data measured along a flat plate. But nevertheless Spalding et al. neglected the parameter $M_\infty$ in Fig. 12 and put a mean curve through the data, namely

$$F_{R\theta} = \frac{\mu_w}{\mu_\infty} (1 + B)^{3/8}$$

(2.22)
resulting in a correlation demonstrated in Fig. 13 where the drawn line represents the universal skin friction law from Fig. 11. As one must expect there is again a well-defined Mach number effect in Fig. 13 so that eq. (2.22) needs an improvement taking into account the parameter $M_\infty$. This fact becomes even more obvious in Fig. 14 where Spalding et al.'s calculation method is compared with experimental data by plotting $c_f/c_{f0}$ against $2F/c_{f0}$. Their theory predicts an increase of the skin friction reduction with increasing Mach number for the same injection rate and Reynolds number. All measurements and theories indicate an opposite tendency. Introducing a Reynolds analogy as defined by eq. (2.14) Spalding et al. extended their calculation method based on the similarity parameters

\[
\frac{c_f}{c_{f0}} \quad \text{vs} \quad \frac{2F}{c_{f0}}
\]

Fig. 15. The reduction of the Stanton number due to air injection (comparison of Spalding et al.'s theory with experimental data).
\( F_C, F_{R_\theta}, \text{ and } F_{R_x} \) to include the influence of heat transfer. Analysing experimental data they found with a considerable scatter that the Reynolds analogy factor \( s \) can be set approximately to 0.825 independent of Mach number, injection rate and heat transfer at the wall. Consequently Fig. 11 and Fig. 13 can be used to predict Stanton numbers \( c_h \) when the ordinate \( F_{CC_f} \) is replaced by \( sc_h F_C \).

The prediction of Spalding's theory for the Stanton number \( c_h \) is shown on Fig. 15 and compared with experimental data plotting \( c_h/c_{R_\theta} \) against \( F/c_{R_\theta} \) for \( R_x = 10^6 \). As for the skin friction coefficient the Mach number effect is predicted wrongly, especially for incompressible flow \( (M = 0) \) where the discrepancy between theory and experiment is considerable. Even for a Mach number of 4 the agreement between theory and experiment is not very good.

Spalding et al.'s method should not be used for incompressible flow and flows at Mach numbers smaller than two. For these cases the prediction of the skin friction and Stanton number and especially their variation with the Mach number is questionable and disagrees with every other theory and, what is more significant, with all measured data for incompressible flow. The advantage of Spalding et al.'s theory is the relatively quick calculation of skin friction and heat transfer coefficients so that it might be interesting for quick engineering calculations. The last point might justify an improvement of the present calculation concept to include in more detail the Mach number effects, especially the influence of \( M_\infty \) on \( F_{R_\theta} \).

2.1.5. Boundary layer transformations

2.1.5.1. The present status of boundary layer transformations. A completely different approach to solve the problem of compressible turbulent boundary layers with air injection is the application of a boundary layer transformation. All the transformations between compressible and incompressible boundary layers are inspired by the work of Dorodnitsyn (70) and Howarth (71) who showed that a simple change of the coordinate \( y \) normal to the wall into

\[
y^* = \int_0^y \frac{\theta}{\theta^*} \, dy
\]

(2.23)

could reduce the compressible laminar boundary layer equation to an quasi-incompressible form. This work was extended by Illingworth (72) and Stewartson (73) who showed that a transformation of both coordinate
could produce an exact mathematical correspondence between two-dimensional laminar incompressible and compressible boundary layers for the case of zero heat transfer, unit Prandtl number and assuming that the viscosity can be taken proportional to temperature. In this work it was assumed that the stream functions in the two flows are invariant under the transformation.

In the case of a turbulent layer it is, of course, impossible to get a complete mathematical transformation since the mechanism determining the shear stress distribution through the boundary layer cannot be expressed in mathematical terms at present so that only a theory–experiment compromise remains. A number of authors (Spence\textsuperscript{(74)}, Mager\textsuperscript{(75)}, Burggraf\textsuperscript{(76)}) have extended the transformation concept for laminar flows to turbulent flows. These authors assumed that the stream functions remain invariant. Despite these restrictive assumptions all these authors did succeed in getting good agreement with experiment for some of the boundary layer properties.

A notable advance in the transformation technique was made by Coles\textsuperscript{(62)} who assumed that the stream functions in the two corresponding flows were not equal. This work was further extended and rationalized by Crocco\textsuperscript{(77)}. Obviously the fact that the stream functions are not equal in the two flows increases the mathematical complexity of the transformation but reduces the number of arbitrary assumptions. It is true that their substructure hypothesis does represent a compromise with the unknown nature of turbulence, but this hypothesis may be readily adopted as the theory of turbulence is developed.

Coles\textsuperscript{(62)} succeeded in reducing all available experimental data for the skin friction coefficient of compressible boundary layers to corresponding incompressible values. Recently Baronti and Libby\textsuperscript{(78)} investigated the correctness of a point-to-point mapping of compressible turbulent boundary layer flow into its corresponding incompressible form using Coles’s transformation. They did not use Coles’s substructure hypothesis but preferred to employ the assumption that the Reynolds number associated with the laminar sublayer is the same for compressible as well as incompressible flow. Baronti and Libby succeeded in correlating the compressible boundary layer profile (up to a Mach number of $M = 6$) with the law of the wall for the region where it normally holds for incompressible flow. Discrepancies arose, however, when they attempted to transform the outside part of the compressible boundary layer profile into the velocity defect law. They attributed these discrepancies to favourable pressure gradients, but it is quite possible that the reasons are deeper and might
be related to the sublayer concept itself or a failure of the boundary layer transformation in this region.

Rosenbaum\(^{(79)}\) tried to extend Coles’s transformation concept to compressible boundary layers with heat transfer and mass diffusion. He assumed that the incompressible flow field is completely known when the boundary layer is divided in the usual manner into a laminar sublayer, a region where the law of the wall holds and another where the velocity defect law is valid. The corresponding compressible flow field can be easily determined from the incompressible one with the help of Coles’s transformation. However, contrary to Crocco\(^{(77)}\) Rosenbaum used the now known compressible flow field to solve numerically the equations of conservation of energy and species by an implicit finite difference solution technique. This approach is questionable because his calculations are based on the same incompressible boundary layer with zero heat transfer and zero mass diffusion whether, or not, heat transfer and mass diffusion are present in the corresponding compressible flow. This approach postulates that the compressible boundary layer with heat transfer and mass diffusion can be reduced completely to a corresponding incompressible flow with zero heat transfer and zero mass diffusion. This has not been proved. In this connection one must remember that Rosenbaum used Coles’s transformation which is strictly valid only for the case of zero heat transfer. Despite these doubtful assumptions Rosenbaum obtained reasonable agreement between experimental data and his theoretical values for some boundary layer characteristics.

2.1.5.2. Coles’s transformation. It will increase the understanding of the transformation for transpired turbulent boundary layers if Coles’s transformation for the solid flat plate case is discussed first. Coles’s transformation is valid for the case of zero pressure gradient and zero heat transfer. The idea of this transformation is to establish a correspondence between the compressible boundary layer described by the equations (2.4) to (2.6) and a simpler form of flow—in general an incompressible flow described by the equations (2.38) to (2.40)—which has been more thoroughly investigated experimentally as well as analysed theoretically. In order to reduce the compressible boundary layer equations (2.4) to (2.6) to the corresponding equations (2.38) to (2.40) for incompressible flow Coles introduced three transformation parameters \(\sigma\), \(\eta\) and \(\xi\) which are derived from the stream functions \(\psi\) and \(\psi^*\) where starred symbols refer in general to the

\[\text{Crocro introduced transformation rules for the enthalpy distribution as well in order to transform the energy equation.}\]
transformed stage which is identical here with incompressible flow. The stream functions are defined in usual way by

\[ \varrho u = \frac{\partial \psi}{\partial y}, \quad \varrho v = -\frac{\partial \psi}{\partial x} \]  

(2.24)

\[ \varrho^* u^* = \frac{\partial \psi^*}{\partial y^*}, \quad \varrho^* v^* = -\frac{\partial \psi^*}{\partial x^*}. \]  

(2.25)

The connection between the stream functions \( \psi \) and \( \psi^* \) are the transformation parameters \( \sigma, \eta \) and \( \xi \) which are in general functions of the coordinates \( x \) and \( y \). They are defined by

\[ \sigma(x, y) = \frac{\psi^*(x^*, y^*)}{\psi(x, y)} = \sigma(x) \]  

(2.26)

\[ \eta(x, y) = \frac{\varrho^*}{\varrho} \frac{\partial \psi^*}{\partial y^*} = \eta(x) \]  

(2.27)

and

\[ \xi(x, y) = \frac{dx^*}{dx} = \xi(x). \]  

(2.28)

Coles showed that the three functions \( \sigma, \eta \) and \( \xi \) were functions of \( x \) only, or rather that \( \sigma, x^* \) and \( \partial y^*/\varrho \partial y \) are independent of \( y \) when second order derivatives are neglected. It can be proved that an exact correspondence for the inertia and pressure gradient term of the compressible and incompressible flow can be established mathematically when the transformation parameters are introduced into eq. (2.4) and (2.5). The weak point of the transformation is the deduction that the shear stress terms can be transformed as well, which cannot be proved in general because of the unknown shear stress distribution in physical as well as mathematical terms. Assuming that the shear stress distribution of the compressible boundary layer can be reduced to a corresponding one in incompressible flow and postulating that a boundary layer with constant pressure transforms into an incompressible one with zero pressure gradient, the transformation leads to two expressions connecting the transformation parameters \( \sigma, \eta \) and \( \xi \) with the usual boundary layer variables, namely

\[ \frac{\sigma}{\eta} = \frac{u^*_w}{u_w} = \frac{u^*}{u} = \text{constant} \]  

(2.29)

and

\[ \xi = \sigma \eta \frac{\varrho^* \mu_w}{\varrho^* \mu^*} \left\{ 1 + \frac{2}{c_f} \left[ \frac{1}{\sigma} \frac{d\sigma}{dx} \left( \Theta + \frac{\psi_w}{\varrho_w u_w} \right) \right] \right\} \]  

(2.30)
with \( \psi_w = 0 \) for the case of zero injection. Moreover, transformation rules can be derived for the interesting boundary layer characteristics like for example \( c_f, \Theta, R_x \) connecting them with corresponding quantities in incompressible flow.

In order to determine the three unknown quantities \( \sigma, \eta \) and \( \xi \) one needs a third relationship, which Coles derived from his substructure hypothesis. He assumed that there exists a certain turbulent substructure corresponding to a well-defined Reynolds number \( R_s \) within the boundary layer which is independent of compressibility effects when the density and viscosity are evaluated at a suitably chosen mean temperature \( T_m \). This is a rather nebulous approach, and the physical interpretation of this concept rather doubtful, even when the substructure hypothesis is considered in its more rationalized form as presented by Crocco. The substructure concept is mainly justified by its application since it succeeds in reducing all available data for the skin friction coefficients measured along solid flat plates to an equivalent incompressible value. This correlation must be regarded as excellent considering the published raw data obtained in different wind tunnels with their unavoidable irregularities of tunnel conditions and the effect of tripping devices of the experimental set-up, for example.

2.1.5.3. Jeromin's transformation. Jeromin extended Coles's transformation concept for compressible turbulent boundary layers on solid surfaces to layers with air injection. The present transformation concept is practically identical to that of Coles since it is based on the same definitions for the transformation parameters \( \sigma, \eta \) and \( \xi \) so that the equations (2.29) and (2.30) hold here as well. The difference occurs in the determination of the third relation connecting the boundary layer parameters with the usual boundary layer variables.

For the case of fluid injection this relation follows from the definition equations for the stream functions (2.24) and (2.25). The stream functions \( \psi \) and \( \psi^* \) must hold all through the boundary layer and hence at the wall. Thus

\[
\psi_w^* = -\int_0^{x^*} \rho^* v_w^* \, dx^*
\]

and

\[
\psi_w = -\int_0^x \rho v_w \, dx
\]

when the virtual origin of the flow is set to zero for both the incompressible and compressible flow, which is no restriction to the problem. One obtains
assuming that \( q^* v^*_w \) and \( q_w v_w \) are constant along the wall,

\[
\sigma = \frac{\psi^*_w}{\psi_w} = \frac{\Theta^* v_w x^*}{q_w v_w x^*} \tag{2.31}
\]

making use of eq. (2.26). Equation (2.31) is the required third relationship defining, together with (2.29) and (2.30), all three transformation parameters. The disadvantage of defining \( \sigma \) by eq. (2.31) is the fact that it does not include the case of zero injection. For the case of zero injection eq. (2.31) becomes indeterminate even when one tries to approach this limiting case by assuming very small injection rates \( q_w^* v_w^* \).

Eliminating \( \sigma \) from (2.29) to (2.31) leads to

\[
\frac{d\sigma}{dx} = \frac{\sigma^2 - A \sigma}{\sigma x^* + 2AF x^* + 2A \frac{\Theta^*}{c_f^*}} \tag{2.32}
\]

with the constant

\[
A = \frac{u^*_w}{u_\infty} \frac{\Theta^* \mu^*}{\Theta^* \mu_w} \frac{q_w v_w}{q^* v^*_w}.
\]

Once \( \sigma \) is known the other parameters \( \eta \) and \( \xi \) can be determined from (2.29) and (2.30).

It is the principle of the present transformation that the properties for incompressible flow like \( u_\infty^* \), \( q^* \), \( \mu^* \) and \( F^* \), which must be known to integrate eq. (2.32), can be chosen arbitrarily without affecting the transformation. Moreover, as Crocco pointed out one obtains the same transformation if the incompressible fluid chosen is a liquid or a gas. After establishing a starting condition \( \sigma_B = f(R_{xB}, F, F^*) \) and empirical relations defining \( c_f^* = f(R_x^*, F^*) \) and \( \Theta^* = f(R_x^*, F^*) \) equation (2.32) has been integrated leading to graphs \( \sigma = f(R_x) \) with the parameters \( F \) and \( F^* \). These graphs are the key of the transformation and have been used to apply the transformation to measured compressible boundary layer profiles with air injection.

The main advantage of a boundary layer transformation is the extension of the range of application of well-established theories for incompressible flow to compressible flow. As Stevenson\(^{61}\) and McQuaid\(^{228}\) have shown,\(^\dagger\) Stevenson's law of the wall

\[
\frac{2u^*_x}{v^*_w} \left\{ \sqrt{1 + \frac{v^*_w u^*_x}{u^*_x^2}} - 1 \right\} = \frac{1}{\kappa} \ln \frac{y^* u^*_x}{v^*} + C \tag{2.33}
\]

\(^\dagger\) For details see Section 2.2.
with $\alpha = 0.410$ and $C = 5.00^*$ for transpired incompressible boundary layers is highly successful for determining the skin friction coefficient for measured boundary layer profiles. Equation (2.33) holds for the fully turbulent region of the boundary layer so that it can be considered as a formula defining $u_r^*$ and hence $c_f^*$ for a given boundary layer profile, for example by solving (2.33) for measured $u^*$ and $y^*$. This concept can be used to determine the skin friction coefficient for the compressible boundary layer by transforming the quantities $u$ and $y$ into their corresponding incompressible form and evaluating $u_r^*$ from eq. (2.33). The friction velocity $u_r^*$ is constant in the region where the law of the wall holds. Its value will be chosen to determine the skin friction coefficient $c_f^*$ since it gives the best agreement with Stevenson's law. The compressible skin friction coefficient follows from the incompressible one, again using the transformation. Skin friction coefficients determined in this manner agreed extremely well with those obtained from the momentum equation as Jeromin$^{(8)}$ has shown for compressible boundary layers at $M = 2.5$ and $M = 3.5$ and three different injection rates.

The success of the present transformation can be illustrated best by plots in Stevenson's coordinates. It was postulated that an incompressible flow with an injection rate $F^*$ can be chosen arbitrarily. This postulate has been checked on Fig. 16 for two representative compressible boundary layers measured by Jeromin$^{(23,66)}$ at a Mach number of 3.5 for two different injection rates. These two profiles were chosen at random and can be considered as being representative for all other profiles measured by Jeromin. The fully turbulent part of the compressible transpired boundary layer profiles has been reduced to Stevenson's law of the wall for incompressible flow independent which injection rate $F^*$ was chosen. Only injection rates $F^*$ up to $3 \times 10^{-3}$ can be checked for the time being since the transformation parameter cannot be evaluated accurately enough for higher injection rates (for details see Jeromin$^{(8)}$). In Fig. 16 the collapse for the fully turbulent region is excellent for all injection rates $F^*$ chosen. The skin friction coefficient evaluated for these compressible profiles from Stevenson's law of the wall is independent of the choice of $F^*$. A collapse for the outer part of the compressible profile cannot be expected from such a plot since the transformed flows correspond to different Reynolds numbers $R_x^*$ and $R_y^*$. Moreover, one would expect such a reduction only by

\* The values for $\alpha$ and $C$ are mean values since there is a considerable amount of disagreement in the literature regarding their actual values. Coles's values $\alpha = 0.410$ and $C = 5.0$ were mainly chosen because they are good mean values (see Black and Sarnecki$^{(60)}$).
FIG. 16. Comparison between transformed compressible turbulent boundary layer profiles with air injection and Stevenson's law of the wall using Jeromin's transformation. Mach number 3.5.

plotting the outer region of the profile in appropriate coordinates such as

$$\frac{u_{*}}{u_{t}} = \frac{\delta_{l}}{\delta_{*}}$$

(2.34)

of the velocity defect law for example. This must remain the subject of a separate investigation and is mentioned here only for reasons of completeness.

Figure 17 is representative for all compressible boundary layers measured at Mach numbers of 2.5 and 3.5 with three injection rates $F$. The fully turbulent part of the compressible boundary layer profile is again completely reduced to a corresponding incompressible one. This collapse is not as good for the highest injection rate of $M = 3.5$. But this profile is nearly separating (blown-off) so that it is rather surprising that it could be collapsed at all. The injection rate for the incompressible flow was chosen to be $F^* = 2 \times 10^{-3}$. The best possible skin friction coefficient was fitted again to these profiles by solving Stevenson's law of the wall for the transformed compressible flow.
There seems to be a big discrepancy between the profiles measured at a Mach number of 2.5 and those obtained for \( M = 3.5 \), especially for the outer part of the profile. This applies for all the diagrams concerning Stevenson's law of the wall. The reason for this discrepancy is the different Reynolds number for the transformed compressible flow. This number is about twice as high for the profiles obtained for a Mach number of 2.5 than for those at \( M = 3.5 \). McQuaid showed that the profiles which he measured in incompressible flow shift to the right (the level of \( y^* u^* / \nu^* \) increases) for increasing Reynolds number \( R^*_\theta \) when the profile is plotted in Stevenson's coordinates. At the same time the overall length of the fully turbulent region increases with increasing \( R^*_\theta \). These are exactly the same features as found for the transformed flow as one can see for example on Fig. 17.

Experimental data for the skin friction coefficient and values for \( c_f \) evaluated from Stevenson's law in connection with the present boundary layer transformation are compared on Fig. 18 with each other. The agreement between theory and experiment is excellent for both Mach numbers.
investigated. One can say that the present transformation together with Stevenson's law gives the best theoretical prediction for the skin friction coefficient for the time being. Moreover, Jeromin has shown that the present boundary layer transformation succeeds in reducing the momentum thickness of compressible boundary layers to corresponding incompressible values. These excellent results so far obtained for the fully turbulent region of the boundary layer profile, the skin friction coefficient and the momentum thickness are a good starting-point for an extension of the present boundary layer concept to include the case of pressure gradient and heat transfer. The present transformation has been formally extended by Jeromin considering these effects. These extensions lead to differential
expressions similar to eq. (2.32) which cannot be solved at present because of lack of information about the variation of \(c_j^*\) and \(\Theta^*\) with \(R_x^*, F^*\), \(dp^*/dx^*\) and \(T_w^*/T_{\infty}^*\) so that the application of a transformation for compressible turbulent boundary layers with pressure gradients and heat transfer must remains a future project.

As it has already been mentioned the weak point of a turbulent boundary layer transformation is the correspondence of the shear stresses in both flows. It is a matter of speculation whether such a correspondence exists or not, since the mechanism causing the distribution of the shear stress through the boundary layer is at present unknown and it might be possible that the nature of turbulence in compressible and incompressible flow is different, or, in other words, that the shear stress distributions are governed by completely different physical laws in both kinds of flow which cannot be reduced in each other. This cannot be decided at present and the near future appears unlikely to provide a solution. At the moment the only possible evidence or counter-evidence can be deduced from experimental results and the validity of this proof is also limited. Anyway, investigations are in progress to check the validity of the present transformation more generally by trying to transform measured shear stress profiles of compressible boundary layers with air injection. Moreover, the validity of the transformation will be checked in the region where the velocity defect law holds.

2.1.6. The effective displacement thickness

It is sometimes necessary to evaluate the flow field around a body. For this purpose one must know the displacement thickness of the boundary layer. Mann\(^{81}\) and Hayasi\(^{82}\) have pointed out that the effect of fluid injection on the external flow should include an additional term considering the injected mass flow. The effective displacement thickness for a transpired boundary layer along a flat plate becomes

\[
\Delta^* = \delta^* + \int_0^x \frac{\rho_w u_w}{\rho_{\infty} u_{\infty}} \, dx \quad (2.35)
\]

(see ref. 81). Hayasi derived a similar formula for the effective displacement thickness of a compressible axisymmetrical transpired boundary layer.
2.2. Incompressible Flow

The incompressible turbulent boundary layer with air injection is not only interesting in its own right but becomes even more important in connection with the boundary layer transformations used to describe boundary layer phenomena in compressible flow (see Coles\(^{(62)}\) and Jeromin\(^{(8, 23)}\)). In the following section the status of research in incompressible flow is summarized and discussed to present a starting-point for further investigations in this field.

2.2.1. Experimental investigations

According to its importance quite a number of experimental investigations concentrate on the incompressible boundary layer with air injection. The most interesting facts about these experimental studies are summarized in Table 2.\(^{\dagger}\) It contains the necessary information about the experimental set-up, injected gas, covered range for the injection parameter \(F = v_w/u_\infty\), temperature, Reynolds number and measured quantities.

The flat plate configuration was used by most investigators. Early experimental studies were carried out by Eckert, Diaguila and Patrick\(^{(83)}\) and Brunk\(^{(84)}\). They must be considered as preliminary investigations which are obsolete now. More thorough experimental studies of the incompressible boundary layer along a flat plate with air injection were carried out by Mickley and Davies\(^{(48)}\), Romanenko and Kharchenko\(^{(85, 86)}\), Wuest\(^{(87)}\) and McQuaid\(^{(88, 89)}\). Mickley and Davies's data does not seem to be very reliable because of a slight adverse pressure gradient during their experiments which was not considered in the data reduction. This adverse pressure gradient leads to a slightly bigger reduction of the skin friction coefficient due to air injection than found by other investigations as one can see on Fig. 1 where their data are plotted in the appropriate coordinates of \(c_f/c_{f0}\) against \(2F/c_{f0}\). Mickley and Davies's data (including velocity profiles) are published in tabulated forms so that they can be easily used for further theoretical analyses. Moreover, there is still the possibility that the data can be corrected for the slight adverse pressure gradient.

Romanenko and Kharchenko\(^{(85)}\) published data for the skin friction coefficient and the Stanton number for transpired boundary layers. With

\(^{\dagger}\) A few experimental analyses of the incompressible turbulent boundary layer with distributed air injection could not be considered in Table 2. To the author's knowledge a couple of experimental studies have been carried out at M.I.T. Unfortunately the experimental data are only published in form of theses and not elsewhere, so that they are not accessible to everybody.
their experimental set-up it was possible to vary the wall temperature considerably so that reliable heat transfer measurements were possible. The values for the skin friction coefficient are plotted in Fig. 1 and show a slightly smaller reduction of the skin friction coefficient due to injection than found by Mickley and Davies. Romanenko and Kharchenko's Stanton number measurements are summarized in Fig. 2. These are the only data at present available showing the Stanton number reduction due to injection for incompressible flow. They have been used for comparisons with theories.

Wuest\(^{87}\) has measured temperature and velocity profiles of turbulent boundary layers along a perforated surface. He injected cold and warm air into the main stream so that the influence of fluid injection on the Stanton number and the boundary layer growth as a whole can be studied from his investigation.

McQuaid's\(^{28, 88}\) experimental investigation is certainly the most thorough and systematical one. His measurements of boundary layer profiles and the resulting evaluation of the skin friction coefficients and the other boundary layer characteristics must be considered as the most reliable information at present available. His values for the skin friction coefficient are plotted on Fig. 1 and have been used to represent the experimental results for comparisons with theories. The Reynolds number influence in Fig. 1 is such that the data shifts in the direction of the arrow for \( F = \text{constant} \). Since practically all data lie on one line, one can deduce from Fig. 1 that the Reynolds number influence seems to be negligible. One can say that the Reynolds number effect is fully described in this figure by the skin friction coefficient \( c_{f0} \) for the case of zero injection. McQuaid's data have been published in form of tables so that they are available for further investigations (see ref. 88).

Moreover, McQuaid\(^{28}\) investigated experimentally the influence of favourable and slight adverse pressure gradients on the development of incompressible transpired boundary layers. In a second investigation Romanenko and Kharchenko\(^{88}\) measured the local skin friction coefficient in the presence of favourable and adverse pressure gradients for transpired turbulent boundary layers. Both investigations must be considered as early attempts to include pressure gradients in the boundary layer analysis so that they are just mentioned here without going into the details. Certainly much more systematical measurements, especially for the case of adverse pressure gradients, are necessary before final conclusions can be drawn.

There are two other experimental analyses of the incompressible turbu-
<table>
<thead>
<tr>
<th>Investigator(s)</th>
<th>Geometry</th>
<th>Injected gas</th>
<th>$F \cdot 10^3$</th>
<th>Temp. (°K)</th>
<th>Measured quantity</th>
<th>Reynolds number</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>Eckert, Diaguila and Patrick$^{(63)}$ 1955</td>
<td>flat plate</td>
<td>air</td>
<td>0 to 17</td>
<td>room temp.</td>
<td>velocity profiles, local skin friction coefficient</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Brunk$^{(64)}$ 1957</td>
<td>flat plate</td>
<td>air</td>
<td>0 to 16</td>
<td>$T_{\infty} = 375$</td>
<td>local skin friction coefficient, wall temperature measurements</td>
<td></td>
<td>results very unreliable because of experimental difficulties; compressibility effects</td>
</tr>
<tr>
<td>Mickley and Davies$^{(44)}$ 1957</td>
<td>flat plate</td>
<td>air</td>
<td>0 to 10</td>
<td>room temp.</td>
<td>velocity profiles, local skin friction coefficient, boundary layer parameters</td>
<td>$0.4 \leq \frac{d}{R_{x} \times 10^{-5}} \leq 30$</td>
<td></td>
</tr>
<tr>
<td>Tewfick, Eckert and Jurewicz$^{(83)}$ 1961</td>
<td>circular cylinder</td>
<td>air</td>
<td>0 to 2.9</td>
<td>room temp.</td>
<td>local Stanton number and skin friction coefficient</td>
<td>$1 \leq \frac{d}{R_{x} \times 10^{-5}} \leq 1.03$ \ $2 \leq \frac{d}{R_{x} \times 10^{-5}} \leq 15$</td>
<td>$x = \text{distance along surface of cylinder, } D = \text{diameter of cylinder}$</td>
</tr>
<tr>
<td>Romanenko and Kharchenko(^{(88)}) 1963</td>
<td>flat plate</td>
<td>air helium CO(_2) Freon-12</td>
<td>0 to 7</td>
<td>450 (\leq T_\infty \leq 550)</td>
<td>local skin friction coefficient and Stanton number</td>
<td>(1 \leq R \times 10^{-5}) [per metre] (\leq 5)</td>
<td></td>
</tr>
<tr>
<td>---</td>
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<td></td>
</tr>
<tr>
<td>Tewfick(^{(90)}) 1963</td>
<td>circular cylinder</td>
<td>air</td>
<td>0 to 3·1</td>
<td>room temp.</td>
<td>local and average skin friction coefficient, shear stress distribution</td>
<td>(0.5 \leq R \times 10^{-4}) (\leq 6.0) [per metre]</td>
<td></td>
</tr>
<tr>
<td>Romanenko and Kharchenko(^{(88)}) 1963</td>
<td>flat plate</td>
<td>air CO(_2) Freon-12</td>
<td>0 to 7·6</td>
<td>local skin friction coefficient</td>
<td>(1 \leq R \times 10^{-5}) [per metre] (\leq 5)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Stevenson(^{(91)}) 1964</td>
<td>circular cylinder</td>
<td>air</td>
<td>0 to 5·7</td>
<td>room temp.</td>
<td>local skin friction coefficient, boundary layer profiles</td>
<td>(R = 1 \times 10^6) [per metre]</td>
<td></td>
</tr>
</tbody>
</table>

\(x = \) distance from the tip of the cylinder

\(^{1)}\) the investigation includes the effect of favourable and adverse pressure gradients
<table>
<thead>
<tr>
<th>Investigator(s)</th>
<th>Geometry</th>
<th>Injected gas</th>
<th>$F \times 10^3$</th>
<th>Temp (°K)</th>
<th>Measured quantity</th>
<th>Reynolds number</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wuest$^{(87)}$ 1965</td>
<td>flat plate</td>
<td>air</td>
<td>0 to 7-6</td>
<td>room temp.</td>
<td>velocity and temperature profiles</td>
<td>$1 \leq R_s \times 10^{-5}$</td>
<td>the investigation includes the effect of favourable and adverse pressure gradients</td>
</tr>
<tr>
<td>Olson and Eckert$^{(92)}$ 1965</td>
<td>circular cylinder</td>
<td>air</td>
<td>0 to 5-8</td>
<td>room temp.</td>
<td>velocity profiles, local skin friction coefficient, shear stress profiles</td>
<td>$R = 0.92 \times 10^5$ and $2.69 \times 10^5$ (per metre)</td>
<td></td>
</tr>
<tr>
<td>McQuaid$^{(18, 88)}$ 1966</td>
<td>flat plate</td>
<td>air</td>
<td>0 to 8</td>
<td>room temp.</td>
<td>velocity profiles, local skin friction coefficients, shear stress profiles, boundary layer parameters</td>
<td>$3.5 \leq R_s \times 10^{-5}$</td>
<td></td>
</tr>
</tbody>
</table>
lent boundary layer with air injection along a flat plate, namely by Goodwin\(^{(51)}\) and Smith\(^{(52)}\). Unfortunately the data of both investigations have been published to the author's knowledge only in form of Ph.D. theses which are not accessible to everybody. Consequently these data have not been considered here.

The boundary layer characteristics of an axisymmetric flow with fluid injection were the subject of experimental investigations by Tewfick \textit{et al.}\(^{(88, 90)}\), Stevenson\(^{(91)}\) and Olson and Eckert\(^{(92)}\). The latter paper is especially important in that not only are the usual boundary layer parameters presented, but also shear stress and eddy diffusivity distributions through the boundary layer, evaluated from the measured velocity profiles, are included. How reliable these data for axisymmetric flow are as checks for theories predicting quantities for two-dimensional flow along a flat plate cannot be estimated at present. Consequently these measurements are not considered in Figs. 1 and 2 since these diagrams are mainly used to represent experimental data for comparisons with available theories. Practically all these theories are derived for the flat plate geometry.

\subsection*{2.2.2. Simple film and sublayer theories}

A simple film theory has been proposed by Mickley \textit{et al.}\(^{(93)}\) which replaces the actual boundary layer by a film of constant thickness, neglecting any variations in the flow direction as compared with those across the film. There are now much more sophisticated theories available based on more realistic boundary layer models so that Mickley \textit{et al.}'s approach will be just mentioned here.

Turcotte\(^{(50)}\) introduced a sublayer theory, assuming that the effect of air injection is restricted to the sublayer region only. He divides the boundary layer into a laminar sublayer, a buffer layer and an outer region. Molecular transport processes predominate in the laminar sublayer, turbulent transport phenomena occur mainly in the outer region whereas the buffer layer is characterized by both processes. The viscosity distribution in the buffer layer was approximated by Turcotte by the empirical expression

$$\frac{s_M}{\nu} = \sinh^2 \left( \frac{b Re'}{13.89} \right)$$

with

$$Re' = \frac{y u_t}{\nu}$$

which was first introduced by Rannie\(^{(94)}\) for the case of zero injection for which the factor \(b\) is equal to unity. Introducing the hypothesis that the
shear stress at the outer edge of the buffer layer is unaffected by fluid injection leads to a simple expression for the skin friction, namely

$$\frac{\tau_w}{\tau_{w0}} = \exp \left[ -6.94 \frac{v_w}{u_{r0}} (1 + \sqrt{\frac{\tau_{w0}}{\tau_w}}) \right]$$

(2.37)

where the subscript 0 refers to zero injection. This assumption postulates that the mechanism of turbulent momentum transfer is so strong compared with laminar transfer that the effect of fluid injection can only affect the region where the laminar transfer predominates whereas the quantities in the outer region of the boundary layer are unchanged and identical to those for the case of zero injection.

Despite these restrictive assumptions the shear stress at the wall is predicted in the right order of magnitude. But this result is fortuitous as Leadon has proved by plotting velocity and shear stress profiles with and without injection. They indicate that the outer region of the boundary layer is strongly affected by fluid injection similarly to the case of adverse gradients. This result shows that much more complicated boundary layer models must be used to predict the interesting quantities of the boundary layer with reasonable accuracy.

### 2.2.3. Theories based on mixing length concept

#### 2.2.3.1. General remarks

The incompressible boundary layer is described by the following equations:

**continuity equation**

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

(2.38)

**momentum equation**

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{dp}{dx} + \frac{1}{\rho} \frac{\partial \tau}{\partial y}$$

(2.39)

with

$$\tau = (\mu + \varepsilon_M) \frac{\partial u}{\partial y}, \quad \varepsilon_M = -\frac{\rho u'v'}{\partial u/\partial y};$$

**energy equation**

$$\rho u \frac{\partial h}{\partial x} + \rho v \frac{\partial h}{\partial y} = u \frac{dp}{dx} + \frac{\partial q}{\partial y} + \tau \frac{\partial u}{\partial y}$$

(2.40)
These equations have been solved by a number of authors for the case of constant pressure with the help of Prandtl's mixing length concept for the shear stress distribution

\[ \tau = \rho \kappa^2 y^2 \left( \frac{\partial u}{\partial y} \right)^2. \]

Two theories have been already discussed, namely the theoretical analyses by Rubesin and Dorrance and Dore which are applicable for compressible as well as incompressible flow. It was shown earlier on Figs. 8 and 10 that Dorrance and Dore's predictions for the skin friction coefficient and the Stanton number agree extremely well with experimental data whereas Rubesin's theory can only be recommended for compressible flow (see Figs. 5 and 9).

In this connection a third theoretical analysis by Clarke, Menkes and Libby must be mentioned which is very similar to those by Rubesin and Dorrance and Dore. Based on a dimensional analysis Clarke et al. assumed that the velocity distribution throughout the boundary layer can be described by

\[ \frac{u}{u_r} = f \left[ \frac{u_r y}{v}, \frac{v_r}{u_r} \right] = A + B \ln \frac{u_r y}{v} + \frac{1}{4 \rho^2} \frac{v_r}{u_r} \ln^2 \frac{u_r y}{v}, \quad (2.41) \]

with the constants \( A, B \) and \( \kappa \) to be determined from experimental data. The value for these constants were chosen such that they agree with those found in conjunction with the universal log-law

\[ \frac{u}{u_r} = \frac{1}{\kappa} \ln \frac{y u_r}{v} + C, \quad (2.42) \]

for boundary layers along impermeable surfaces, hence setting \( A = 1/\kappa, \) \( B = C \) and neglecting completely the influence of fluid injection on these constants. The disadvantage of this simplification can be seen on Fig. 19 where Clarke et al.'s theory is compared with experimental data. The agreement between theory and experiment is poor and can certainly be improved taking into consideration recent experimental investigations and evaluating \( A, B \) and \( \kappa \) accordingly. It will be found from such an analysis
of experimental data that especially $A$ and $B$ are functions of the injection mass flow and that consequently alternative expressions for eq. (2.41) must be introduced in order to solve the boundary layer equations (2.38) and (2.39) and to make $A$, $B$ and $\kappa$ to real constants. Such dimensionless

$$\frac{c_f}{c_{f0}}$$

![Graph showing the reduction of the skin friction coefficient due to air injection.](image)

**Fig. 19.** The reduction of the skin friction coefficient due to air injection (comparison of Clarke, Menkes and Libby's theory with experimental data).

laws for the velocity distribution throughout the boundary layer will be discussed in the next sections.

The concept of dividing the boundary layer in the usual manner into a laminar sublayer, a buffer layer and a fully turbulent region was also used by other investigators. Torii and Nishiwaki for example derived
similarly to eq. (2.41) an expression for the velocity distribution, namely

\[
\frac{u}{u_*} = \frac{1}{\kappa} \ln \frac{u_* y}{v} + \frac{v_w}{4u_*} \left( \frac{1}{\kappa} \ln \frac{u_* y}{v} + C \right)^2
\]

which reduces to the law of the wall (2.42) for the case of zero injection with \( v_w = 0 \). The constants \( \kappa \) and \( C \) are assumed to be identical to those for the case of zero injection. As one can see in Fig. 20 the agreement of

Torii and Nishiwaki's dimensionless velocity distribution equations with measured transpired boundary layer profiles is not very satisfactory, when compared, for example, with the agreement obtained by Stevenson's law of the wall for turbulent boundary layers with suction and injection which will be discussed in the next section. But this disagreement, especially with higher injection rates, might again be due to the choice of their constants. Moreover the comparison of Torii and Nishiwaki's universal law with experimental data is based on measurements by Mickley et al. which were subject to errors in measuring the distance normal to the wall. Hence Fig. 20 is not conclusive in all respects and gives only an idea what the boundary layer model looks like. There is certainly still the chance that their theoretical result can be improved taking into consideration more recent experimental studies of the turbulent boundary layer with air injection (for example the investigation by McQuaid) and adjusting their constants accordingly. As one might expect from the insufficient agreement between eq. (2.43) and measured velocity distribution the pre-
diction for the skin friction coefficient is poor and does not agree with experimental data.

Finally Torii and Nishiwaki calculated temperature profiles from their universal velocity profiles and determined the heat transfer coefficient and the Stanton number, assuming a Prandtl number of 0.7 for the laminar sublayer and of 1.0 for the turbulent layer. They obtained reasonable agreement between their Stanton number variation and the experimental results measured by Mickley and Davies\textsuperscript{1} and Romanenko and

\textsuperscript{1} Mickley and Davies concentrated in their investigation only on the influence of air injection on velocity profiles, boundary layer parameters and the skin friction coefficient. They did not publish heat transfer data. The Stanton numbers used by Torii et al.\textsuperscript{2} are calculated from Mickley and Davies's published boundary layer profiles by plotting them in the coordinates of Torii universal laws.
Kharchenko (88). But this agreement does not seem to be very conclusive considering their poor correlations for the skin friction coefficient on which the derivation of the Stanton number relation is based (see Fig. 21). Two more successful calculation methods for boundary layer developments based on dimensionless velocity distributions will be discussed in Section 2.2.5.

2.2.3.2. The law of the wall. Two attempts at extending the law of the wall (eq. (2.42)) to transpired boundary layers have been already discussed. Other concepts for extensions of eq. (2.42) to turbulent boundary layers with suction and injection have been introduced by Black and Sarnecki (80), Stevenson (61) and Tennekes (97), whereby Tennekes’s law is based purely on empirical concepts so that it is just mentioned here only for reason of completeness. Black and Sarnecki’s bilogarithmic law and Stevenson’s law of the wall for transpired boundary layers are very similar and both can be reduced to the law of the wall for boundary layers along impervious walls. Owing to their importance they must be discussed in more detail.

By using mixing length theory Black and Sarnecki (80) showed that the law for the fully turbulent region of the boundary layer is

\[ u_{t}^{2} + v_{w}u = \left( \frac{v_{w}}{2 \kappa} \ln \frac{y}{d} \right) \]  

--- the so-called bilogarithmic law --- with \( \kappa \) the mixing length constant and \( d \) the constant of integration. In order to obtain a relationship similar to eq. (2.42) Black and Sarnecki rearranged (2.44) writing

\[ \frac{u}{u_{t}} = \frac{u_{t}}{v_{w}} (\lambda^{2} - 1) + \frac{\lambda}{\kappa} \ln \frac{u_{t}y}{v} + \frac{1}{4\kappa^{2}} \frac{v_{w}}{u_{t}} \left( \ln \frac{u_{t}y}{v} \right)^{2} \]  

(2.44.1)

with

\[ \lambda = -\frac{1}{2\kappa} \ln \frac{u_{t} \cdot d}{v} \]  

(2.45)

As one can easily see, eq. (2.44.1) reduces with \( v_{w} = 0 \) to the law of the wall for boundary layers along solid walls.

A particular form of the bilogarithmic law was proposed by Stevenson (81). Rearranging eq. (2.44.1) to

\[ \frac{2u_{t}}{v_{w}} \left( 1 + \frac{v_{w}u}{u_{t}^{2}} - 1 \right) = \frac{1}{\kappa} \ln \frac{u_{t}y}{v} + \frac{2u_{t}}{v_{w}} (\lambda - 1) \]
and analysing available experimental data, Stevenson found that the term 
\( \frac{2u_t}{v_w} (\lambda - 1) \) is not affected by injection and is identical to that for boundary
layers along solid walls; hence it follows that

\[
\frac{2u_t}{v_w} \left\{ \sqrt{1 + \frac{v_w u_t}{u_e^2}} - 1 \right\} = \frac{1}{x} \ln \frac{u_e y}{v} + C
\]  

(2.46)

which is equivalent to accepting

\[
\lambda = \frac{C}{2} \frac{v_w}{u_t} + 1.
\]  

(2.45.1)

Equation (2.46) will be called Stevenson's law. The right-hand sides of the
equations (2.42) and (2.46) are identical; even the constants \( x \) and \( C \) are
the same.

The main advantage of Stevenson's law compared with the bilogarithmic
law is in its usefulness in correlating all boundary layers with suction or
injection or even those on solid walls for the case of constant pressure on
one straight line for the fully turbulent region when choosing the coordi-
nates accordingly. McQuaid (28) evaluated skin friction coefficients quite
successfully applying Stevenson's law to his measured boundary layer
profiles with air injection. He interpreted eq. (2.46) as a relationship defin-
ing \( u_t \) and hence \( c_f \) for a given profile by solving Stevenson's law for
measured \( u \) and \( y \) with the help of a computer. Such a calculation results
in the determination of a pseudo friction velocity \( u'_t \) which is only constant
within the fully turbulent region of the boundary layer where

\[
u' = u = u_e \sqrt{c_f/2}.
\]

The value for \( u_t \) was chosen to determine the skin friction coefficient
since it gives the best agreement with Stevenson’s law. Skin friction
coefficients determined in this manner from Stevenson’s law agreed ex-
tremely well with those calculated from the momentum integral equation
as McQuaid (28) and Stevenson (61) have shown.

A typical plot for transpired boundary layers in Stevenson’s coordinates
for McQuaid's (28) velocity profiles is shown in Fig. 22 for \( x = 0.435 \) and
\( C = 5.45 \). The collapse of the measured profiles on Stevenson’s law is
excellent for the fully turbulent region for all injection rates considered
in Fig. 22. In this connection it might be worth mentioning that velocity
profiles measured along solid flat plates would give the same collapse since
McQuaid's data are corrected for displacement effects and turbulent fluctuation.

Fig. 22. Stevenson's (61) law of the wall in comparison with McQuaid's (88) experimental data after McQuaid.
the ordinate reduces to \( u/u_z \) for \( v_w = 0 \). It should be noted that the skin friction coefficients for the profiles in Fig. 22 are those which give the best agreement with Stevenson’s law and that they are not independently measured but checked by the momentum integral equation. As the Reynolds number increases the profiles shift upwards along the straight line in Fig. 22. At the same time the overall length of the fully turbulent region increases.

Analogous to the well-known Clauser charts, Stevenson’s recommendation for the parameter \( \lambda \) (eq. 2.45) allows the calculation of a chart (a so-called inner region profile family) for each injection rate enabling one to determine quickly the value for the skin friction coefficient which gives the best agreement with Stevenson’s law. One example is shown in Fig. 23 for \( v_w/u_\infty = 3.2 \times 10^{-8} \) together with four experimentally determined velocity profiles (different \( R_o \)) at this injection rate.

![Fig. 23. Comparison of measured velocity profiles for \( F = 3.2 \times 10^{-8} \) with Stevenson’s inner region profile family (skin friction chart) after McQuaid.](image)

Black and Sarnecki’s bilogarithmic law on the other hand is not so handy for the determination of the skin friction coefficient. The parameter \( \lambda \) can be evaluated from experimental velocity profiles writing the bilogarithmic law in the form

\[
\frac{u}{u_\infty} - y_i^2 = (n_i^2 - p_i^2) - 2n_i y_i
\]  
(2.44.2)
with

\[ y_t = \frac{1}{2\kappa} \sqrt{\frac{v_w}{u_\infty}} \ln \frac{u_\infty y}{\nu} \]

\[ n_t = -\frac{1}{2\kappa} \sqrt{\frac{v_w}{u_\infty}} \ln \frac{u_\infty d}{\nu} \]

\[ p_t = \sqrt{\frac{u^2}{v_w u_\infty}}. \]

Plotting now measured profiles in coordinates \( \left( \frac{u}{u_\infty} - y_t^2 \right) \) against \( y_t \), the profile should lie on a straight line in the region where (2.44.2) holds. The values for \( n_t \) and \( p_t \) can be easily determined from the intercept of the profiles with the axis and the slope of the straight lines. Once \( n_t \) and \( p_t \) are known the parameter \( \lambda \) and the skin friction coefficient determined in this manner should agree with those evaluated from Stevenson's law. But as McQuaid (98) has proved, this agreement depends to a certain extent on the choice of the constants \( \kappa \) and \( C \). If the choice of the constants is not correct, then negative skin friction coefficients may be predicted when the bilogarithmic law is compared with measured velocity profiles, a result not supported by the momentum integral equation. In particular, negative values for \( c_f \) will be predicted especially at large injection rates if the value of \( \kappa \) is too small. Analysing very carefully his own and Mickley and Davies's corrected data† McQuaid (98) proved that the prediction of the bilogarithmic law and Stevenson's law for \( c_f \) agree with each other when \( \kappa \) is accepted to be 0.435 and \( C = 5.9 \), both constants independent of the injection rate. The values for \( \kappa \) and \( C \) lie certainly within the spread of values found in the literature for boundary layers along impervious surfaces:

\[
0.385 \leq \kappa \leq 0.470
\]

\[
4.07 \leq C \leq 7.15.
\]

The constants do not seem to be universal, at least not for wind tunnel experiments. They seem to be dependent on experimental conditions. Several possible reasons for the spread of the constants have been proposed like effects of differences in free stream turbulence level (Landweber (100)), methods of boundary layer transitions (Coles (83)) and the magnitude of quasi-periodic variations of boundary layer properties (Thompson (101)).

† These data have been corrected taking into consideration the slight adverse pressure gradient which was apparently present during their experiments.
Finally one can say that Stevenson's law is the best way at present available for the determination of the skin friction coefficient for measured boundary layer profiles in the presence of air injection. The constants \( \alpha \) and \( C \) most probably depend on the experimental set-up and should be determined first for boundary layers along solid walls where several possibilities are available for the determination of the skin friction coefficient with reasonable accuracy. Once \( c_f \) is known the constants can then easily be determined from the law of the wall for solid walls and then used to determine the skin friction coefficient for transpired boundary layers.

2.2.4. The velocity defect law

The mean velocity distribution in the outer region of turbulent boundary layer profiles was the subject of a number of investigations in recent years. Most of them can be traced back to at least one of the following boundary layer models:

(i) Coles's\(^{62}\) law of the wake;
(ii) Sarnecki's\(^{109}\) intermittency hypothesis;
(iii) Clauser's\(^{110}\), Townsend's\(^{111}\), Mellor and Gibson's\(^{112}\) or Libby, Baronti and Napolitano's\(^{113}\) constant eddy viscosity analyses.

Tennekes\(^{97}\), Stevenson\(^{102}\), Mickley et al.\(^{103}\) and McQuaid\(^{28, 104}\) extended these concepts and established velocity defect laws for turbulent boundary layers with suction and injection by plotting

\[
\frac{u_{\infty} - u}{u_*} = f\left(\frac{y}{\delta}\right)
\]

or using slightly modified coordinates.

Tennekes's approach is again the most direct one in adjusting the velocity defect law for turbulent boundary layers along solid flat plates to the case of transpired boundary layers using the same plots as for the solid plate case and adjusting the constants empirically taking into consideration the effect of air injection.

Mickley and Smith's\(^{103}\) analysis is based on Clauser's\(^{110}\) model for turbulent boundary layers assuming an outer region of constant eddy viscosity "floating" on a more complicated fully turbulent region to which it is coupled only loosely. Computing shear stress profiles from measured constant pressure boundary layer profiles, Mickley and Smith found that the shear stress rose rapidly from its wall value, reached a well-defined maximum in the region \( 0.1 < y/\delta < 0.2 \) and decreased to zero at \( y/\delta = 1 \) as demonstrated qualitatively in Fig. 24 in comparison with the case of zero.
injection. Taking into consideration the shear stress distribution for transparent boundary layers in connection with Clauser’s boundary layer model, Mickley and Smith deduced that the maximum shear stress now play the dominating part in the presence of injection instead of the wall shear stress for the solid wall case. Thus they modified Coles’s\textsuperscript{14} well-known empirical correlation for boundary layers along impervious walls

\[
\frac{u_{\infty} - u}{u_{*}} = -\frac{1}{\kappa} \ln \frac{y}{\delta} + \frac{\pi(x)}{\kappa} \left[ 2 - w\left(\frac{y}{\delta}\right) \right] \quad (2.47.1)
\]

to an alternative expression replacing \(u_{*}\) by a friction velocity \(u_{*}^*\) based on the maximum shear stress, hence

\[
\frac{u_{\infty} - u}{u_{*}^*} = -\frac{1}{\kappa} \ln \frac{y}{\delta} + \frac{\pi(x)}{\kappa} \left[ 2 - w\left(\frac{y}{\delta}\right) \right]. \quad (2.48)
\]

Equation (2.48) will be called Mickley and Smith’s velocity defect law. Assuming now that the outer part of the boundary layer is unaffected by fluid injection, \(\kappa, \pi(x)\) and \(w(y/\delta)\) should be unaffected by fluid injection and hence they should be the same as for the solid wall case, namely \(\kappa = 0.41\) and \(\pi(x) = 0.55\) as proposed by Coles\textsuperscript{82}. Plotting their experi-
mental velocity profiles in coordinates according to eq. (2.48) Mickley and Smith obtained excellent agreement between their velocity defect law and the measured profile as demonstrated in Fig. 25 for two injection rates $F$

![Graph showing velocity defect law](image)

**Fig. 25.** The velocity defect law after Mickley and Smith.\(^{108}\)
and different Reynolds numbers $R_0$. For the higher injection rate, the agreement is not as good, but it is well within the precision of the data. One must remember in this connection that one has to differentiate measured velocity profiles in order to determine the shear stress profile and hence the wanted maximum shear stress. Such calculation process is subject to unavoidable errors so that the slight scatter in Fig. 25 must be also seen in this light.

Another velocity defect law for transpired boundary layers, which overcomes the problem of evaluating the maximum shear stress, was proposed by Stevenson\(^{102}\). He calculated his friction velocity $u_\tau$ from the shear stress at the wall as for boundary layers along impervious walls. Based on a dimensional similarity Stevenson derived his velocity defect law for the outer region of the boundary layer, which is of the form

$$\frac{2u_\tau}{v_\infty} \left\{ \sqrt{1 + \frac{v_\infty u_\infty}{u_\tau^2}} - \sqrt{1 + \frac{v_\infty u}{u_\tau^2}} \right\} = F\left(\frac{y}{\delta}\right)$$

(2.49)

where $F(y/\delta)$ is a universal function for the case of constant pressure independent of the injection mass flow. His analysis is based on the assumption that there exists within the boundary layer an overlap region where both the velocity defect law as well as his law of the wall (eq. (2.46)) are valid.

All the velocity defect law plots are very sensitive in respect to the definition of the boundary layer thickness. Townsend introduced a velocity defect law for the case of zero injection which is of the form

$$\frac{u_\infty - u}{u_\tau} = f\left(\frac{y}{\delta_0}\right)$$

where $\delta_0$ is defined as the value at which $\frac{u_\infty - u}{u_\tau} = 1$. Similarly Stevenson defined his characteristic boundary layer thickness by

$$\frac{2u_\tau}{v_\infty} \left\{ \sqrt{1 + \frac{v_\infty u_\infty}{u_\tau^2}} - \sqrt{1 + \frac{v_\infty u}{u_\tau^2}} \right\} = 1.$$ 

If the assumption is right that $F(y/\delta_0)$ represents a universal function in eq. (2.49), experimental velocity profiles should fall on one single curve when

$$\frac{2u_\tau}{v_\infty} \left\{ \sqrt{1 + \frac{v_\infty u_\infty}{u_\tau^2}} - \sqrt{1 + \frac{v_\infty u}{u_\tau^2}} \right\}$$

is plotted against $y/\delta_0$. The result is shown in Fig. 26 for corrected Mickley and Davies's data. There is a slight but not systematic scatter but nevertheless a mean curve can be put through the data so that $F(y/\delta_0)$ seems to
be an universal function independent of $v_w$ and $u_r$ for the case of zero pressure gradient within the accuracy of the measurements.

Based on his velocity defect law Stevenson calculated shear stress distributions through the boundary layer which agreed reasonably well with those calculated from measured boundary layer profiles. Moreover, the velocity distribution now known throughout the boundary layer from

![Graph](image-url)

**Fig. 26. The velocity defect curve after Stevenson.**

his law of the wall and his velocity defect law, Stevenson was able to determine the momentum thickness and so the Reynolds numbers $R_0$ or $R_x$ as functions of the skin friction coefficient and the injection rate $F$. The result is shown in comparison with experimental data in Fig. 27 plotting $c_f/c_{f0}$ against $2F/c_{f0}$ for $R_x = 10^6$. The agreement with experimental data is excellent.

Tennekes\(^{(103)}\) presented a formal connection between Mickley and Smith's and Stevenson's velocity defect laws. It cannot be decided at present which concept is better, since the difference between both laws is of the order of 6 per cent, which is about the same order of magnitude of unavoidable
experimental errors, so that the discrepancy between both concepts is at present only of academic interest. Both velocity defect laws are based on rather simple boundary layer models and it cannot be deduced at present which simulates the real physical behaviour of the turbulent boundary

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All the velocity defect laws are only valid in its present form for the case of zero pressure gradient. McQuaid\(^{(28, 104)}\) introduced a more general concept for the velocity distribution in the outer part of the turbulent boundary layer profile which applies strictly only to equilibrium turbulent boundary layers.\(^{1}\) But it was also found that it holds for the outer part

\(^{1}\) For exact equilibrium the shear stress must be constant. This condition is practically fulfilled when the local value for \(c_f\) is about the same as that obtained by taking a mean over several boundary layer thicknesses upstream. Hence the velocity profile cannot be
of near-equilibrium layers, in particular for the constant pressure layer with distributed injection \( (v_w = \text{constant along the plate}) \).

Extending Clauser's concept McQuaid wrote the velocity defect law in the form

\[
\frac{u - u_\infty}{u_*} = f\left(\frac{y}{\delta}\right) \tag{2.50}
\]

with

\[
u_*^2 = u_\infty^2 \frac{d\theta}{dx} = \left(1 + \frac{2}{H}\right) \frac{\delta^*}{\theta} \frac{dp}{dx} \pm \nu_w u_\infty + u_\infty^2 \tag{2.51}
\]

now including pressure gradients. He showed that this unique velocity defect law not only works for normal boundary layers along solid plates but also for boundary layers in adverse and favourable pressure gradients and what is most important in this connection for boundary layers with air injection. An excellent collapse with the defect curves for zero transpiration and zero pressure gradient was obtained for all these profiles as illustrated in Figs. 29 and 30. The zero transpiration curves are determined from Thompson's profile family. In order to obtain the likely spread of the constant pressure velocity defect plot in the coordinates of eq. (2.50), use is made of Thompson's two parameter profile family of mean velocity profiles together with suitably chosen values for \( H \) and \( R_\theta \). The velocity defect profiles thus obtained should well represent the spread of constant pressure results. Hence

\[
H = 1.37 \quad R_\theta = 3.2 \times 10^3
\]

and

\[
H = 1.26 \quad R_\theta = 5 \times 10^4
\]

have been chosen which can be accepted as reasonable extremes.

Stevenson's injection profiles for zero pressure gradients (Fig. 29) and even McQuaid's profiles with distributed air injection in a region of an adverse pressure gradient (Fig. 30) are reduced with reasonable accuracy to the solid wall case and lie within the spread of the two extreme velocity defect curves. Similar agreement was obtained for Mickley and Davies's and McQuaid's profiles as well as for profiles measured by McQuaid in the presence of favourable pressure gradients. In view of the collapse in Figs. 29 and 30, McQuaid's concept for the velocity defect law can be recommended for boundary layer calculations in the presence of pressure described by the velocity defect law for cases where \( c_f \) changes rapidly so that the local value differs considerably from an integrated value over a certain distance along the plate.
gradients and distributed air injection. One must remember in this connection that McQuaid's velocity defect law is the only calculation method at present available to include pressure gradients and air injection. In principle it should also work for boundary layers with suction.

Fig. 29. Stevenson's velocity profiles in McQuaid's velocity defect law coordinates (after McQuaid).

\[ H = 1.26, \quad R_e = 5 \times 10^4 \]
\[ H = 1.37, \quad R_e = 3.2 \times 10^8 \]

Thompson's profile family.

Fig. 30. McQuaid's velocity defect law in comparison with McQuaid's velocity profiles measured in the presence of air injection and an adverse pressure gradient.

\[ H = 1.26, \quad R_e = 5 \times 10^4 \]
\[ H = 1.37, \quad R_e = 3.2 \times 10^8 \]

Thompson's profile family.
2.2.5. Boundary layer development calculation methods

Most of the theories discussed so far concentrate on one particular part of the profile or the most interesting parameters of the boundary layer like skin friction coefficient, for example, but do not allow the calculation of the velocity and temperature profile throughout the boundary layer and hence the boundary layer growth calculations along flat plates or arbitrarily shaped surfaces. There are investigations which overcome this problem, namely the papers by McQuaid and by Torii et al. The incompressible boundary layer profiles of a complete boundary layer development along a permeable flat plate have to be calculated first for both theories. With the boundary layer profile known the wanted boundary layer parameters can be easily determined.

2.2.5.1. Torii et al.'s concept. Torii et al. start with the boundary layer equations in the integral form considering the boundary conditions

\[ y = 0: \quad u = 0, \quad v = v_w = \text{constant} \]
\[ y = \delta: \quad u = u_\infty \]

so that (2.38) and (2.39) can be written in the form

\[ \frac{v}{u_\infty} = \frac{v_w}{u_\infty} - \frac{\partial}{\partial x} \int_0^y \frac{u}{u_\infty} dy \]  \hspace{1cm} (2.38.1) \]

\[ \frac{\partial}{\partial x} \int_0^y \frac{u^2}{u_\infty^2} dy - \frac{u}{u_\infty} \frac{\partial}{\partial x} \int_0^y \frac{u}{u_\infty} dy + \frac{v_w}{u_\infty} \frac{u}{u_\infty} - \frac{\tau}{\rho u_\infty^2} - \frac{\tau_w}{\rho u_\infty^2}. \]  \hspace{1cm} (2.39.1) \]

In order to solve both equations the velocity and shear stress distribution throughout the boundary layer must be known. Torii et al. found from experimental analyses that the velocity boundary layer profiles can be approximated by the dimensionless universal function

\[ \frac{u}{u_\infty} = f \left( \frac{y}{\delta}, F, c_f \right) \]  \hspace{1cm} (2.52) \]

with

\[ \frac{\partial}{\partial x} f \left( \frac{y}{\delta}, F, c_f \right) = 0 \]

when weak Reynolds effects on \( f \) are ignored.

Based on the fact that nearly all viscous and turbulent dissipation of energy takes place within \( 0 \leq y \leq 0.01 \) and that the wall influences vanishes
at a certain distance from the wall, Torii et al. introduced a new model for the shear stress distribution throughout the boundary layer, namely

\[ \frac{\tau}{\tau_w} = \Phi \cdot \psi \]  

(2.53)

where \( \Phi \) is a function representing only the influence of the wall, and the function \( \psi \) representing the rate of reduction of the wall influence with increasing distance from the wall. Both functions are considered to be independent of each other. It is assumed that the function \( \psi \) is expressed by the same function as for turbulent boundary layers along solid flat plates. The function \( \psi \) is presented by Torii in tabulated form and was determined for the case of zero injection assuming that the velocity profile can be expressed by the universal dimensionless form

\[ \frac{u}{u_\infty} = f\left(\frac{y}{\delta}\right) \]

suppressing Reynolds number effects.

The shear stress distribution in the vicinity of the wall follows from eq. (2.8) so that

\[ \Phi = 1 + \frac{F}{c_f} \frac{u}{u_\infty} \]  

(2.54)

Once \( \Phi \) and \( \psi \) are known the equations (2.38.1) and (2.39.1) can be solved by the method of successive approximations choosing first some values for \( F \) and \( c_f \). With the help of the tabulated function \( f_0 = f(y/\delta, F = 0) \) a new function \( f \) can be calculated from the equation (2.38.1) and (2.39.1) for the chosen parameters \( F \) and \( c_f \). If the calculated value \( f \) is not equal to \( f_0 \), \( f \) is used as the second approximation. This process will be repeated until the difference between \( f \) and \( f_0 \) is less than 0.1 per cent. Thus, the velocity profiles are obtained as functions of the parameters \( y/\delta \), \( F \) and \( c_f \).

The agreement of Torii's analysis with measured velocity profiles is excellent and is illustrated in Fig. 31 where \( u/u_\infty \) is plotted against \( y/\delta^* \). Even the prediction for the shear stress distribution throughout the boundary layer must be considered as splendid as shown in Fig. 32.

Despite these encouraging results the prediction for the skin friction coefficient is not as good as one might expect. The result of Torii's calculation method for the reduction of the skin friction coefficient due to injection is smaller than found by experiment as demonstrated in Fig. 33. The reason for this discrepancy is most probably the fact that Torii's analysis
Fig. 31. Velocity profiles in turbulent boundary layers with air injection (after Torii et al. [107]).

Fig. 32. Shear stress distribution in turbulent boundary layers with air injection (after Torii et al. [107]).
concentrates much more on the outer part of the turbulent boundary layer and not so much on the inner part and the wall region which defines the skin friction coefficient by the slope of the velocity profile at the wall.

Finally Torii et al. determined temperature profiles and Stanton numbers from a heat balance making the following assumptions:

(i) heat transfer by radiation can be neglected;
(ii) the longitudinal heat transfer along the wall is negligible compared with the heat transfer normal to the wall;
(iii) the physical properties of the fluid are nearly constant.

FIG. 33. The reduction of the skin friction coefficient due to air injection (comparison of Torii et al.'s calculation method with experimental data).
The prediction for the Stanton number agrees extremely well with experimental data as one can see on Fig. 34. The agreement between theory and experiment is also excellent for the temperature profiles.

Torii et al.'s calculation method can be improved taking into consideration the Reynolds number effects on the velocity profile. Moreover, the boundary layer distribution close to the wall should be considered in more detail.

2.2.5.2. McQuaid's concept. (i) Zero pressure gradient. McQuaid's analysis is based on Sarnecki's assumption that the observed departure of the velocity distribution from the law of the wall in the outer
part of the turbulent boundary layer profile can be explained in terms of an intermittency relation for the mean velocity. Thus the overall time-mean velocity in the intermittent region will be expressed as the sum of the mean velocity over the time the flow is turbulent (called $u_T$) and that over which the flow is potential (called $u_P$) whereby it is assumed that the potential velocity $u_P$ is identical with the free stream velocity $u_\infty$. Thus the overall time-mean velocity can be written as

$$\frac{u}{u_\infty} = \gamma \frac{u_T}{u_\infty} + (1 - \gamma)$$  \hspace{1cm} (2.55)

\[\text{Fig. 35. Comparison of McQuaid's intermittency hypothesis with experiment (after McQuaid\textsuperscript{106}).}\]
where the intermittency factor $\gamma$ is defined as the fraction of the total time for which the flow is turbulent. McQuaid determined the intermittency factor from this experimental data in connection with the overall time-mean velocity equation since the ratio $u/u_\infty$ is known from his experimental data and the ratio $u_T/u_\infty$ can easily be calculated from Stevenson's law of the wall. He found that the intermittency factor distribution through the boundary layer is a unique function unaffected by air injection and

![Image of velocity profile model](image_url)

**Fig. 36.** Velocity profile model (after McQuaid[106]).

is the same as for solid flat plates as shown on Fig. 35 for $F = 0$, $F = 0.0032$ and $F = 0.008$. For his further calculations McQuaid used the mean curve put through his experimental data as indicated in Fig. 35. This mean curve is compared with alternative intermittency distributions proposed by other authors in the last diagram of Fig. 35. As one can see McQuaid's intermittency distribution curve lies within the range of uncertainty of other investigations.

Based on eq. (2.55) McQuaid proposed an overall velocity profile as shown on Fig. 36 which consists of
(i) a viscous sublayer for which

$$\frac{u}{u_\tau} = \frac{u_\tau}{v_w} \left( e^{-\frac{v_w y}{u_\tau}} - 1 \right)$$  \hspace{1cm} (2.56)

as given by Black and Sarnecki\(^{(60)}\) holds;

(ii) a fully turbulent region in which Stevenson's law of the wall (eq. (2.46)) is valid;

(iii) an outer region where the intermittency hypothesis (eq. (2.55)) holds.

Blending regions between these three layers are neglected. It is assumed that the sublayer law and the law of the wall each hold on either side of the junction point between the two laws. It is generally accepted that the errors introduced by the neglect of blending regions are insignificant especially on the integral profile parameters. In this connection it might be worth mentioning that the existing experimental data are not reliable enough to derive laws for the blending region or even to verify the existence of the sublayer law.

The overall velocity profile for constant pressure boundary layers can therefore be expressed in terms of so-called profile families, namely

$$\frac{u}{u_\infty} = f \left( \frac{y}{\delta_s}, c_f, R_\delta, F \right)$$  \hspace{1cm} (2.57)

where \(\delta_s\) is defined as twice the distance from the wall to the position at which \(y = 0.5\) and hence \(R_\delta = \frac{u_\infty \delta_s}{v}\). The profile family relation (2.57) can be transformed into

$$\frac{u}{u_\infty} = g \left( \frac{y}{\theta}, H, R_\theta, F \right)$$  \hspace{1cm} (2.58)

with the compatibility condition

$$c_f = c_f(H, R_\theta, F)$$  \hspace{1cm} (2.59)

and

$$R_\delta = R_\delta(H, R_\theta, F)$$  \hspace{1cm} (2.60)

The two relations (2.59) and (2.60) can be presented in form of graphs or stored in a computer for quick interpolation calculations. In order to obtain comparisons between overall profiles and experiment for given \(F, H\) and \(R_\theta\), the predicted profile from the family can be found by double interpolation in both kinds of graphs which represent (2.59) and (2.60). The profile can then easily be determined from eq. (2.58) and integrated.
to obtain $\theta$ so that the profiles can be plotted in more usual coordinates $\frac{u}{u_\infty}$ against $\frac{y}{\theta}$.

Conclusive checks of the profile family can be made only with experimental data which have not been used to derive the intermittency curve in Fig. 35. Hence the profile family concept is compared with velocity profiles measured by Mickley and Davies\(^{(46)}\) and Stevenson\(^{(91)}\) in Figs. 37 and 38. The agreement must be considered as excellent.

A boundary layer calculation method can now be derived from the profile family concept. All one needs to start the calculation is an auxiliary equation for the determination of the shape factor development which can be based either on the kinetic energy equation, the moment-of-momentum equation or the entrainment equation. As Thompson\(^{(118)}\) has shown the overall dissipation in the boundary layer is likely to be strongly affected by the change in conditions at the wall, so that the solid surface correlation for the dissipation integral might not be suitable for use with
transpired boundary layers. However, he assumed that the entrainment process should not be so influenced by suction or injection. Consequently McQuaid checked Head's\(^{(117)}\) auxiliary equation for the calculation of \(H\) and found that the prediction of Head's entrainment method for \(H\) is in satisfactory agreement with experimental data. Once \(H\) and \(R_e\) are known the values for the corresponding \(c_f\) and \(R_{\delta*}\) follow immediately from the graphs so that the velocity profile can be calculated introducing the unique function for the intermittency distribution and Stevenson's law into the time-mean velocity profile relation. The boundary layer parameters \(H\) and \(R_e\) can then be calculated for this profile so that the procedure determining \(c_f\) and \(R_{\delta*}\) from the graphs can start again. In this manner a whole boundary layer development can be calculated stepwise.

(ii) Pressure gradients. McQuaid has extended his calculation methods to flows with pressure gradients. The results are of acceptable accuracy for conditions where the law of the wall for constant pressure boundary layers is valid over a significant part of the inner profile. The investigation
concentrates on three kinds of layers which are characterized by the pressure parameter

\[
\Delta_l = \frac{\nu}{(u'^2 + v'^w)^{3/4}} \left( \frac{1}{\bar{p}} \right) \frac{dp}{dx}
\]  

(2.61)

Two layers were investigated in the presence of moderate adverse pressure gradients with \(0 < \Delta_l < 0.0037\) and moderate favourable pressure gradients with \(-0.0017 < \Delta_l < 0\). Moreover, the influence of strong favourable pressure gradients with \(\Delta_l \approx -0.0048\) on the boundary layer development has been analysed experimentally.
Thompson\textsuperscript{(101)} has shown for solid surface boundary layers that a fairly wide range of favourable as well as adverse pressure gradients exists for which the law of the wall for constant pressure boundary layers holds and gives acceptable agreement with experiment. Inspired by this result

\[ u^+ = 150 \text{ L.O.F. JEROMIN} \]

[Graph and equation]

McQuaid analysed his measured boundary layer profiles with moderate pressure gradients. He evaluated the skin friction coefficient \( c_f \) for these profiles from Stevenson's law of the wall using the same constants as for the zero pressure gradient case. This method of determining the skin friction coefficient from the law of the wall has been already described in more detail in previous sections. The fully turbulent part of the profiles is well

\[ \text{Fig. 40. Comparison of Stevenson's inner region profile family with McQuaid's experimental profiles in favourable pressure gradient (after McQuaid\textsuperscript{(100)}.} \]
described by Stevenson's law for the case of moderate pressure gradients as one can see in Fig. 39 for adverse and in Fig. 40 for favourable pressure gradients. For adverse pressure gradients the fully turbulent region decreases and for favourable pressure gradients it increases compared with the case of zero pressure gradient. It must be pointed out here that the values for $c_f$ used in Figs. 39 and 40 are those which give the best fit with the experimental profile.

The values for $c_f$ determined from Stevenson's law are then used with the momentum integral equation to calculate the momentum thickness development. The agreement between the calculated and experimental growth of the momentum thickness is excellent so that one seems to be justified in assuming that the unmodified Stevenson's law also holds for layers with moderate pressure gradients.

Moreover, McQuaid showed that the intermittency distribution is not affected by moderate pressure gradients and it is the same as for the zero pressure gradient case so that the profile family as discussed in the previ-
ous section can be applied here, too. The overall profile for the adverse pressure gradient case is compared in Fig. 41 with the profile family. The agreement must be considered to be very good.

For strong favourable pressure gradients with $\Delta_i$ in the order of $-0.0048$ the agreement between the measured profiles and Stevenson's law of the wall becomes less satisfactory and the resulting values for the skin friction coefficient giving the best fit are in doubt. Nevertheless, McQuaid used these values for $c_f$ to calculate the momentum growth from the momentum integral equation. The agreement between calculated and experimentally determined values for the momentum thickness is again satisfactory. This result is even more surprising since the injection and pressure gradient term in the momentum integral equation almost balance so that possible errors in $c_f$ would have a significant effect on $d\theta/dx$.

Finally McQuaid showed that the intermittency model is no longer valid in the presence of such strong favourable pressure gradients of the order of $\Delta_i \approx -0.0048$ so that the profile family concept cannot be used here any more for boundary layer calculations. In general, it must be said that the information about transpired boundary layers in the presence of pressure gradients is far too incomplete to derive results which are conclusive in all respects. McQuaid's investigation must be considered as a good starting point for further investigation. His calculation method based on profile families is the only approach at present available to calculate boundary layers with pressure gradients and air injection.

2.2.6. Similarity solutions

Another approach to solve the boundary layer equations, completely different from those already discussed, has been proposed by Eckert and Sparrow \cite{118} and Donovan et al.\cite{119}. They deal with similarity solutions which have the property that the velocity profiles of various locations along the plate are of similar shape. These similarity solutions can be obtained by dealing with ordinary differential equations rather than partial ones. There is certainly a price which must be paid in order to achieve this mathematical simplification, which is that the blowing or suction velocity must vary along the plate. The main contribution of Eckert's and Sparrow's paper is their attempt to explore possible ways in which the theoretical results for the skin friction coefficient derived from a similarity solution are useful for blowing rate distributions other than those necessary to establish a similarity solution.
3. TURBULENT BOUNDARY LAYERS WITH FOREIGN GAS INJECTION

The injection of a foreign gas makes the problem even more complex than for the case of air injection since diffusion effects on the boundary layer development must also be considered. This additional complication means that for a theoretical approach the diffusion equation must also be satisfied. In view of these difficulties only very few approaches have been made to the theoretical study of the turbulent boundary layer with foreign gas injection. Such information would be not only valuable in the present context but would be also interesting as first attempts to solve the problems of ablation cooling. Even transpiration cooling with liquid metals is of technical interest, as a paper by Robinson et al.\(^\text{(120)}\) has shown.

As in all turbulent boundary layer theories all these theoretical approaches are at present subject to a certain empiricism which is expressed for example in the constants of integration to be evaluated from experimental data. Consequently all the available experimental investigations will be briefly summarized before the theoretical analyses of the turbulent boundary layer with foreign gas injection are discussed.

3.1. Experimental investigations

3.1.1. Helium injection

Only a few experimental investigations concentrate on the determination of the details of the boundary layers like velocity, temperature or concentration profiles and the evaluation of the usual boundary layer parameters. Romanenko and Kharchenko\(^\text{(68)}\) measured velocity and temperature profiles in the presence of distributed helium injection whereas the concentration profile was evaluated from a gas mixture formula. Unfortunately all these profiles are not published either in the form of graphs or tables so that they are not accessible for further detailed examinations. Romanenko and Kharchenko only published the reduction of the skin friction coefficient and the Stanton number due to helium injection. The reduction of the local skin friction coefficient and Stanton number is bigger for helium than for air injection as one can see in Figs. 42 and 43. The reason for the bigger reduction is the smaller molecular weight of helium compared with air so that a layer of reduced density and viscosity builds up close to the wall. Helium with its very high heat capacity is a very effective coolant as one can see from Fig. 43.
The measurement of the skin friction coefficient in the presence of helium injection was the subject of only one other investigation, namely by Pappas and Okuno\(^{(31)}\) at Mach numbers of 0.3, 0.7, 3.21, and 4.23. They measured the average skin friction coefficient of a porous cone. Their results are included in Fig. 42 despite the fact that it is doubtful whether such measurements are reasonable checks for theories predicting local skin friction coefficients for the flat plate configuration. But it must be pointed out that their data are the only ones at present available for compressible flow. As for air injection the reduction of the skin friction coefficient decreases with increasing Mach number for \(2F/c_{f0} = \text{constant}\). Comparing Fig. 1 with Fig. 42 it can be deduced that the influence of the Mach number on

\begin{figure}
\centering
\includegraphics[width=\textwidth]{image}
\caption{The reduction of the skin friction coefficient due to helium injection compared with air injection (experimental data).}
\end{figure}

\begin{table}
\centering
\begin{tabular}{|c|c|c|}
\hline
Symbol & M & Geometry \hline
\(\times\) & 0.3 & flat plate \hline
\(\triangle\) & 0.7 & cone \hline
\(\triangle\) & 3.21 & \hline
\(\triangleright\) & 4.23 & Pappas and Okuno \((32)\) \hline
\end{tabular}
\end{table}
the reduction of the skin friction coefficient due to helium injection is not so well defined as for air injection.

The only important parameters of heat transfer studies like Stanton number and recovery factor were studied in the experimental investiga-

![Graph](image-url)

**Fig. 43.** The reduction of the Stanton number due to helium injection compared with air injection (experimental data).

...tions by Leadon and Scott\(^{(24, 25)}\), Scott, Anderson and Elgin\(^{(40)}\) and Pappas and Okuno\(^{(31)}\). Leadon and Scott evaluated the local Stanton number and the recovery factor from measured wall temperatures along a porous flat plate in the presence of helium injection at a Mach number of 3. Their data for the recovery factor suggest that its reduction due to helium injection is bigger than for air injection for the same injection rate. But many
more systematical measurements are necessary before this result can be accepted generally. For example, Scott, Anderson and Elgin[40] found that the reduction of the recovery factor due to helium and air injection is about the same within the accuracy of the measurements. They measured velocity and concentration profiles of compressible turbulent boundary layers with helium injection along a flat plate at a Mach number of 3. The Stanton number was evaluated again from a heat balance. They agree extremely well with those measured by Leadon and Scott at the same Mach number as it can be seen in Fig. 43. Mean curves have been put through Romanenko and Kharchenko’s data at \( M = 0 \) and Leadon and Scott’s and Scott et al.’s data at \( M = 3 \). They will represent the experimental results in the next sections where they are compared with available theoretical analyses.

For reason of completeness Pappas and Okuno’s data for the Stanton number are included in Fig. 43. They measured the local Stanton number and local recovery factor for a cone at Mach numbers of 0.7, 3.67 and 4.35. Their results will be not considered in all respects in the next sections because of the unknown effect of the cone geometry compared with the flat plate configuration. Nevertheless, their data are in good qualitative agreement with the results by Leadon and Scott and Scott et al. for the flat plate geometry.

3.1.2. Nitrogen injection

Bartle and Leadon[27] investigated the influence of nitrogen injection on the heat transfer. Unlike all other investigations discussed so far they obtained an isothermal wall temperature by establishing a particular injection distribution along their porous flat plate considering the variation of the heat transfer in \( x \)-direction. For turbulent boundary layers the injection distribution is suspected to vary approximately as \( x^{-1/5} \). Consequently the chamber underneath the flat plate was compartmented and the flow to the various sections controlled by valves. This arrangement permits the injection distribution to be adjusted to give the desired isothermal wall condition.

All their measurements were carried out at Mach numbers of 2.0 and 3.2. The main subjects of their experimental investigation were boundary layer profiles, the recovery factor and the Stanton number; the latter was evaluated from a heat balance. Their results are presented in the form of graphs and tables so that they are easily accessible for future investigations. As one would expect the effect of nitrogen injection on the boundary layer heat transfer characteristics is very similar to the effect of air injection.
The result for the Stanton number and the recovery factor reduction due to nitrogen injection is shown in Fig. 44, Fig. 3 respectively. The Stanton number reduction agrees qualitatively with that found for air injection. The reduction of the Stanton number decreases with increasing Mach number for constant injection mass flow or better for $F/c_{h0} = \text{constant}$. The reduction of the recovery factor due to injection decreases for increasing Mach number.

3.1.3. Carbon dioxide injection

Only Romanenko and Kharchenko\(^{(85)}\) have investigated the effect of carbon dioxide injection on the development of an incompressible bound-
ary layer along a porous flat plate. Compared with air injection the reduction
of the skin friction coefficient as well as of the Stanton number is smaller
due to the higher molecular weight of carbon dioxide as one can see in
Figs. 45 and 46. The injection of carbon dioxide is about 50 per cent less
effective in reducing skin friction and heat transfer at the wall than air-

Fig. 45. The reduction of the skin friction coefficient due to CO₂ injection
compared with air injection (experimental data).

injection. In order to get the same reduction in heat transfer and skin
friction the injection mass flow of carbon dioxide must be about twice as
high compared with air injection. A study of carbon dioxide injection at
Mach numbers of 0-5, 1-8, 2-5 and 3-5 is currently being made at Cam-
bridge University. Measurements are being made of concentration, veloc-
ity and temperature profiles.
3.1.4. Freon-12 injection

The effect of the injection of gases of relatively high molecular weight on the boundary layer development have been studied by injecting Freon-12 through a porous surface into the main stream. Such experimental investigations have been carried out by Romanenko and Kharchenko. The experimental set-up of Romanenko and Kharchenko was the flat plate. They only concentrated on incompressible flow and measured velocity and temperature profiles. The concentration profile was again evaluated from a gas mixture formula. But only their results for the skin friction coefficient and the Stanton number are published and presented in Figs. 47 and 48. The reduction of the skin friction coefficient and the Stanton number due to Freon-12 injection compared with air injection (experimental data).

Fig. 46. The reduction of the Stanton number due to CO₂ injection compared with air injection (experimental data).
friction coefficient due to Freon-12 injection is considerably smaller than for air injection and even smaller than for carbon dioxide injection. The effect of Freon-12 injection on the Stanton number shows the same tendency.

![Graph](image)

**Fig. 47.** The reduction of the skin friction coefficient due to Freon-12 injection compared with air injection (experimental data).

Pappas and Okuno\(^{31, 32}\) used a porous cone in order to obtain information about the influence of Freon-12 injection on the boundary layer parameters \(c_f\) and \(c_b\). They measured the average skin friction coefficient for the cone at Mach numbers of 0·3, 0·7, 3·21 and 4·23 (see ref. 32). As shown in Fig. 47 their data for approximately incompressible flow (\(M = 0·3\)) show a bigger reduction of the skin friction coefficient due to Freon-12 injection than found by Romanenko and Kharchenko. Whether
this discrepancy is caused by the different experimental set-up of a flat plate and a cone or by the fact that the reduction of local and average skin friction are compared with each other, cannot be decided at present. In this connection it must be noted that all these experimental data are subject to possible errors in the order of at least 10 per cent which have to be considered when results of one investigation are compared with those of other workers or even with available theories. These possible errors increase with increasing injection rate so that the scatter of the data increases with increasing injection rate. As it can be deduced from Fig. 47 the effect of compressibility for constant injection rate is such that the
reduction of the skin friction coefficient due to Freon-12 injection decreases with increasing Mach number.

The scatter of Pappas and Okuno's data for the Stanton number is even bigger than for the skin friction coefficient as shown in Fig. 48. Their values for the Stanton number predict again a bigger reduction for $C_h$ than found by Romanenko and Kharchenko. But one tendency is again clear: that the influence of Freon-12 injection on the reduction of the Stanton number becomes less effective with increasing Mach number.

3.2. The Reference State and Effective Temperature Concept

Because of the complexity of the problem a number of engineering approaches have been proposed which are all more or less based on a reference state concept. Apart from the usual reference state equations for the temperature a second empirical relation must be introduced defining a reference concentration. The major contributions using a reference state concept are the papers by Scott and Knuth and Dershin. Knuth and Dershin make a virtue of necessity because of the rather incomplete information about turbulent boundary layers with foreign gas injection and recommend the reference-state expressions developed by them for transpired laminar boundary layers as a suitable solution to the present problem, at least for the time being. But as they have pointed out it cannot be deduced at present from available experimental data whether this approximation can be justified or not. A much wider range of Reynolds and Mach numbers must be covered as well as the influence of several other foreign gases on the boundary layer behaviour studied before final conclusions can be drawn. For quick engineering calculations this pragmatical approach might be useful but from the scientific point of view it has no future and will certainly become obsolete as the turbulent boundary layer with foreign gas injection becomes much better understood.

The effectiveness concept used by Mager and Divoky, Tewfick, Bartle and Leadon, Knuth and Dershin and Tilford can also be considered as a kind of reference state concept. The basic idea of the effectiveness concept is the definition of an effectiveness parameter for the skin friction and heat transfer coefficient, which manages to correlate all experimental data measured at different Mach numbers, wall temperatures, injection rates and with different injection gases on a single curve by plotting the effectiveness parameter against a suitably chosen coordinate like a temperature ratio or a modified blowing rate parameter. The effectiveness parameter is a function of the wall temperature, free stream temper-
ature, blowing rate, velocity, skin friction and heat transfer coefficient. Because of the extreme empiricism involved in the effectiveness concept these possibilities of simplifying the problem in the way as described above will be just mentioned here without going too much into the details. The contribution of Bartle and Leadon\(^{(121)}\) might be interesting in particular for engineers where it is necessary to estimate very quickly the reduction of heat transfer due to foreign gas injection in the right order of magnitude. They attempted to find a unique relationship between the effectiveness of transpiration cooling (air injection) and mass transfer cooling (foreign gas injection) and proposed an effectiveness parameter

\[
R = \frac{T_w - T_F}{T_{aw0} - T_F} \left( 1 + \frac{c_{PF} F}{c_{P\infty} \rho_0} \frac{q_0}{q} \right)^{-1}.
\]

As Tewfick has pointed out the present definition of the effectiveness has the disadvantage of a singularity for \(T_r = T_{aw0}\), hence for very small cooling rates. Fortunately the engineer is only interested in high cooling rates for the purpose of making the wall temperature considerably different from the adiabatic wall temperature so that the singularity is not of interest for technical applications.

### 3.3. Applications of mixing length theory

More detailed investigations are based on mixing-length theory again and are mainly extensions of the available theories for air injection. The main feature of all these theories is the assumption that the diffusion influence of foreign gas injection is restricted to the sublayer. Only a binary gas mixture is considered in these theories; the primary fluid (air) flowing over the surface represents one component while the injected gas represents the other. All the theories include some integration constants, as do all turbulent boundary layer theories, which have to be evaluated from available experimental data.

All these theories have much in common with each other since they are practically all based on the same boundary layer model so that it is sufficient to discuss only one theory in more detail. The additional effect of foreign gas injection on a theoretical solution of the problem is such that not only the usual boundary layer equations (2.4) to (2.6) have to be satisfied but also simultaneously the diffusion equation

\[
\rho u \frac{\partial C_F}{\partial x} + \rho v \frac{\partial C_F}{\partial y} = \frac{\partial}{\partial y} \left[ \rho(D + \varepsilon_D) \frac{\partial C_F}{\partial y} \right]
\]
with the eddy coefficient of mass diffusion
\[ \varepsilon_D = -\frac{\nu'C'_F}{\partial C_F/\partial y}, \]
the mass fraction of injected fluid \( C_F = q_F/q \) and the diffusion coefficient \( D \).

In the presence of foreign gas injection the energy equation (2.6) has to be modified to
\[
\varrho u \frac{\partial H}{\partial x} + \varrho v \frac{\partial H}{\partial y} = \frac{\partial}{\partial y} \left[ \left( \frac{\mu}{Pr} + \frac{\varepsilon_M}{Pr_T} \right) \frac{\partial H}{\partial y} \right] \\
+ \frac{\partial}{\partial y} \left[ \left( \mu \left( 1 - \frac{1}{Pr} \right) + \varepsilon_M \left( 1 - \frac{1}{Pr_T} \right) \right) \frac{\partial}{\partial y} \left( \frac{\mu^2}{2} \right) \right] \\
+ \frac{\partial}{\partial y} \left[ \left( \frac{\mu}{Sc} \left( 1 - \frac{Sc}{Pr} \right) + \frac{\varepsilon_M}{Sc_T} \left( 1 + \frac{Sc_T}{Pr_T} \right) \right) \left( h_F - h_A \right) \frac{\partial C_F}{\partial y} \right]
\]
(3.3)

(with the turbulent Prandtl number \( Pr_T = \frac{Pr_{CP}}{Pr} \) and the turbulent Schmidt number \( Sc_T = \frac{Sc}{Pr} \)) taking now into consideration the effect of foreign gas injection on the energy balance.

All the theories for the turbulent boundary layer with foreign gas injection are subject to several simplifications of the boundary layer model; the main assumptions are:

(i) the pressure gradient \( dp/dx \) is equal to zero;
(ii) all variations in \( x \)-direction are negligible compared with those in \( y \)-direction (Couette flow);
(iii) the boundary layer is divided in a laminar sublayer and a turbulent outer region without considering a blending region;
(iv) the gases are ideal;
(v) the shear stress distribution through the turbulent region can be described by Prandtl’s mixing length theory;
(vi) a Reynolds analogy is valid between momentum, energy and mass transfer;
(vii) only binary systems are considered consisting of the main stream fluid air and an injected foreign gas.

Based on this rather restrictive boundary layer model several theoretical analyses have been published; the more important studies are by Rubesin and Pappas\(^{(68)}\) for light gas injection, Pappas\(^{(124)}\) for heavy gas injection,
Culick and Ness. Only the theoretical analysis by Ness will be described here in more detail since it is more general than the other and considers compressible as well as incompressible flow.

The simplifications of reducing the boundary layer problem to a Couette flow type is such that the partial differential equations become ordinary ones and uncoupled so that they can be integrated separately which is most important in this connection. In such a calculation process the dependent variables will be related to the coordinates normal to the surface. The streamwise variations (in x-direction) will be obtained introducing the momentum integral equation.

As for the case of air injection the continuity equation reduce for the Couette model to \( \rho v = \rho w v_w = \text{constant} \). Assuming that the molecular transport is dominant in the laminar sublayer the boundary layer equations reduce in this region as follows:

**momentum equation**

\[
\rho v \frac{du}{dy} = \frac{d}{dy} \left( \mu \frac{du}{dy} \right) \tag{3.4}
\]

**energy equation**

\[
\rho v \frac{dH}{dy} = \frac{d}{dy} \left( \frac{\mu}{Pr} \frac{dH}{dy} \right) + \mu \left( 1 - \frac{1}{Pr} \right) \frac{d}{dy} \left( \frac{u^2}{2} \right) + \frac{\mu}{Sc} \left( 1 - \frac{Sc}{Pr} \right) (h_F - h_A) \frac{dC_F}{dy} \tag{3.5}
\]

**diffusion equation**

\[
\rho v \frac{dC_F}{dy} = \frac{d}{dy} \left( \phi D \frac{dC_F}{dy} \right). \tag{3.6}
\]

Similarly the equations for the turbulent region will be obtained neglecting the molecular transport compared with the dominant eddy transport. It follows:

**momentum equation**

\[
\rho v \frac{du}{dy} = \frac{d}{dy} \left( \phi_M \frac{du}{dy} \right) \tag{3.7}
\]

**energy equation**

\[
\rho v \frac{dH}{dy} = \frac{d}{dy} \left( \phi_M \frac{dH}{dy} \right) + \phi_M \left( 1 - \frac{1}{Pr_T} \right) \frac{d}{dy} \left( \frac{u^2}{2} \right) \\
+ \frac{\phi_M}{Sc_T} \left( 1 - \frac{Sc_T}{Pr_T} \right) (h_F - h_A) \frac{dC_F}{dy} \tag{3.8}
\]

**diffusion equation**

\[
\rho v \frac{dC_F}{dy} = \frac{d}{dy} \left( \phi_D \frac{dC_F}{dy} \right). \tag{3.9}
\]
The laminar boundary layer equations satisfy the boundary condition at the wall

\[ y = 0 : u = 0, \quad v = v_w, \quad T = T_w, \quad C_F = C_{FW} \quad (3.10) \]

whereas the equations for the turbulent region are consistent with the boundary condition at the edge of the boundary layer

\[ y = \delta : u = u_\infty, \quad T = T_\infty, \quad C_F = C_{F_\infty} = 0 \quad (3.11) \]

In order to avoid steps in the velocity, temperature and concentration profile at the interface between the laminar sublayer and the turbulent outer region and to make the overlap continuous the following conditions have to be fulfilled at the edge of the laminar sublayer where \( y = s \):

\[ y = s : u_{sL} = u_{sT}, \quad C_{FSL} = C_{FS_T}, \quad T_{sL} = T_{sT} \quad (3.12) \]

\[ \left( \mu \frac{du}{dy} \right)_{sL} = \left( \mu_M \frac{du}{dy} \right)_{sT} \quad (3.13) \]

\[ \left[ -\frac{\lambda}{dy} \text{grad} - \frac{\mu}{M} \text{grad} \right] \left( \frac{dC_F}{dy} \right)_{sL} = \left[ -\frac{\lambda}{dy} \text{grad} - \frac{\mu}{M} \text{grad} \right] \left( \frac{dC_F}{dy} \right)_{sT} \quad (3.14) \]

\[ \left( \frac{\rho D}{dy} \right)_{sL} = \left( \frac{\rho D}{dy} \right)_{sT} \quad (3.15) \]

Before the boundary layer equations for the laminar sublayer can be integrated the Schmidt number distribution must be known. For this purpose Ness used simple polynomials of the form

\[ Sc = x_0 + x_1 C_F + x_2 C_F^2 \quad (3.16) \]

with the constants \( x_i \) to be determined empirically. Such empirical Schmidt number distributions for helium–air mixtures are published for example by Carlson\(^ {184} \).

The variation of the concentration within the sublayer will be obtained by integrating eq. (3.6) twice to yield

\[ F \frac{u_\infty y}{v_\infty} = \left( \frac{T_w}{T_\infty} \right)^{3/4} \ln \left( \frac{C_F - 1}{C_{FW} - 1} \right) \frac{(x_0 + x_1 C_{FW})}{(x_0 + x_1 C_F)} \quad (3.17) \]

whereby the ratio \( (T_w/T_\infty)^{3/4} \) is an approximation for the viscosity ratio \( \mu/\mu_\infty \) for air assuming that the temperature does not vary much through the sublayer.

The velocity distribution throughout the sublayer can be derived by integrating the simplified momentum equation. A relatively simple relation
between the velocity and the concentration will be obtained assuming that 
the Schmidt number of the binary air–helium mixture is a function of the 
concentration only. Considering these simplifications it follows for the 
velocity distribution

\[ 1 + \frac{2F}{c_f} \frac{u}{u_{\infty}} = \left( \frac{C_F + A}{C_{Fw} + A} \right)^{\alpha_1} \left( \frac{C_F + B}{C_{Fw} + B} \right)^{\alpha_2} \left( \frac{C_F - 1}{C_{Fw} - 1} \right)^{\alpha_3} \]  

(3.18)

with the constants \( A, B \) and \( \omega_i \) to be evaluated from empirical relations 
defining the Schmidt number variation through the laminar sublayer.

The concentration and velocity distribution known, the energy equation 
can be integrated yielding to a relation between temperature and velocity 
and hence concentration. Assuming that the specific heat \( c_{PA} \) and \( c_{PF} \) are 
constant one finally obtains after some rather tedious algebraic manipulations

\[ \frac{T}{T_{\infty}} e^\phi - \frac{T_w}{T_{\infty}} = V(1 - e^\phi) - W \frac{u}{u_{\infty}} \left( 1 + \frac{F}{c_f} \frac{u}{u_{\infty}} \right) e^\phi + W \int_0^{u/u_{\infty}} \left( 1 + \frac{2F}{c_f} \frac{u}{u_{\infty}} \right) e^\phi d\left( \frac{u}{u_{\infty}} \right) \]  

(3.19)

with

\[ V = -\frac{c_{P\infty}}{c_{PF}} \frac{Q_w}{F_{Q_w} u_{\infty} h_{\infty}} \]  

(3.20)

\[ W = -\frac{c_{P\infty}}{c_{PF}} \frac{c_f u_{\infty}^2}{2h_{\infty} F} \]  

(3.21)

\[ Q_w = -\lambda_w \left( \frac{dT}{dy} \right)_w + q_w v_w h_w - q_w D_w (h_{Fw} - h_{A w}) \left( \frac{dC_F}{dy} \right)_w \]  

(3.22)

\[ \phi = -\frac{c_{PF}}{c_{PA}} \ln \left( \frac{C_F + E}{C_{Fw} + E} \right)^{\lambda_1} \left( \frac{C_F + F}{C_{Fw} + F} \right)^{\lambda_2} \left( \frac{C_F - 1}{C_{Fw} - 1} \right)^{\lambda_3} \]  

(3.23)

with the constants \( E, F \) and \( \lambda_i \) to be evaluated again from empirical relations 
defining the Schmidt and Prandtl number variation throughout the sublayer.

The concentration, velocity and temperature distribution for the turbu-
lent outer region have to be calculated from the Couette flow type equa-
tions (3.7) to (3.9) taking now into consideration the conditions (3.12) to 
(3.15) at the interface. The calculation procedure parallels, in many re-
spects, the analysis for the laminar sublayer, so that only the final results 
will be given here. Assuming the turbulent Schmidt number constant, the 
relationship

\[ \left( 1 + \frac{2F}{c_f} \frac{u}{u_{\infty}} \right)^{s_{sc}} \]  

(3.24)
between the concentration and the velocity in the turbulent region can be derived. The temperature distribution in the turbulent region is 

$$\frac{T - T_\infty}{T_\infty} = V(1-e^{\phi_T}) + \frac{\partial}{\partial u_f \partial u_\infty} \left(\frac{u_f}{1+c_f u_\infty} - \frac{u}{c_f} \right) + \int_{u_f/u_\infty}^{u/u_\infty} \left(1+\frac{2F}{c_f} \frac{u}{u_\infty}\right) e^{\phi_T} d\left(\frac{u}{u_\infty}\right)$$

(3.25)

with

$$\phi_T = \frac{Pr_T}{Sc_T} \ln \left\{ \left(\frac{C_{F_f} - 1}{C_{F_f} - 1}\right) \left[1+\frac{c_{PF}/c_{PA} - 1}{1+(c_{PF}/c_{PA} - 1)}\right] \right\}.$$ 

(3.26)

Before the Couette flow type equations for the turbulent region can be integrated, expressions have to be defined specifying the velocity at the interface between the sublayer and the fully turbulent region; for example in terms of the skin friction coefficient and the density to take into account compressibility effects. At the present stage of knowledge this relation for \( u_s \) can be only an empirical one; among several alternative expressions (see for example Mickley and Davies\(^{(46)}\) and Rubesin\(^{(55)}\)) Ness used the formula

$$\frac{u_s}{u_\infty} = K \sqrt{\frac{c_f}{2}} \sqrt{\frac{\rho_\infty}{\rho_w}}$$

proposed by Frankl and Voishe\(^{185}\) with the constant \( K \) to be determined from experimental data. This equation was derived for boundary layers along solid surfaces. Before the above-mentioned equation can be used in the present context, it had to be proved in principle by an experimental analysis whether it also applies for transpired boundary layers.

However, before the differential equations describing the velocity, temperature and concentration distributions throughout the boundary layer can be integrated, starting conditions for velocity-concentration relations in the laminar sublayer and in the fully turbulent region have to be derived connecting quantities at the interface with those at the wall and in the free stream. Such an expression can be easily determined by evaluating eq. (3.18) at the interface and eq. (3.24) at the outer edge of the boundary layer and eliminating \( u_s/u_\infty \) from the resulting expression. It will be found

$$\left(\frac{C_{F_f} + A}{C_{F_w} + A}\right)^{w_1} \left(\frac{C_{F_f} + B}{C_{F_w} + B}\right)^{w_2} \left(\frac{C_{F_f} - 1}{C_{F_w} - 1}\right)^{w_2} - \left(1+\frac{2F}{c_f}\right) \left(\frac{C_{F_f} - 1}{C_{F_w} - 1}\right) \right)^{1/Sc_T} = 0.$$ 

(3.18.1)

Equation (3.18.1) represents the condition wanted for the start of the numerical integration. It relates the concentration at the interface to those
at the wall and at the edge of the boundary layer for prescribed values of the turbulent Schmidt number $Sc_f$ and of the factor $2F/c_f$. In this connection it should be pointed out that Ness’s analysis is based on considering the parameter $2F/c_f$ constant along the permeable surface imposing a pointwise variation of the injection mass flow and hence the concentration along the surface. The variation of the concentration can be evaluated for example from eq. (3.18.1). This coupling between the injection parameter $F$ and the skin friction coefficient $c_f$ is certainly a restriction of Ness’s theory. It can be considered only as an good approximation for transpired boundary layers with $F = \text{constant}$ along the surface. But in favour of Ness’s approach it might be worth noting that in most practical applications of transpiration cooling with $F = \text{constant}$ the Reynolds number does not vary significantly along the permeable surface so that the resulting change in $c_f$ is within acceptable limits. Hence Ness’s presumption of $2F/c_f$ equals constant is fulfilled with reasonable accuracy at least for most engineering calculations. Only for the case of constant injection rates and large injection areas in flow direction Ness’s theory should be used with care and one should always have in mind the restrictions of this particular theoretical analysis.

The boundary layer profile completely known, the momentum thickness can be evaluated in order to get the streamwise variation of the most interesting boundary layer parameters in the present context like $c_f$ and $c_h$ from the momentum equation

$$\frac{dR_0}{dx} = \frac{c_f}{2} + F$$ (3.27).

Ness introduced a new starting condition for the integration of the momentum equation by setting at the starting point $C_{F_0} = C_{F_\infty}$ in order to determine in a relatively simple way a starting value for $R_0$. This procedure is analogous to setting $u_x/u_\infty$ equal to unity imposing that only laminar flow exists at the starting point of the integration contrary to the usual starting condition of $\frac{c_f}{2} \to \infty$ and $R_0 \to 0$ for $x \to 0$ presuming an infinite interface velocity at the leading edge.

The prediction of Ness’s theory for the reduction of the skin friction coefficient due to helium injection is shown in Fig. 49 for Mach numbers of 2 and 8 and for $T_w = T_\infty$. The agreement with available experimental data is reasonably good, especially for the Mach number of 2. Experiment and theory show the same tendency that helium injection is about two, times more effective in reducing the skin friction coefficient than air-
injection. Ness’s theory predicts a slightly smaller influence of the Mach number on the reduction of the skin friction coefficient than shown by Pappas and Okuno’s (32) experimental data. But it must be stressed again that neither their experimental investigation nor Ness’s theory are conclusive in all respects, since Pappas and Okuno measured the average skin friction coefficient for a porous cone whereas Ness’s theory predicts local skin friction coefficients for the flat plate geometry using a rather simplified boundary layer model. All one can say is that Ness’s theoretical approach is the best one at present available and that it can be recommended for boundary layer analyses. The result of the theory can be perhaps slightly improved by adjusting the constants of integration according to

![Graph showing skin friction coefficient reduction due to helium injection compared to experimental data.](image)

*Fig. 49. The reduction of the skin friction coefficient due to helium injection (comparison of Ness’s (59) theory with experimental data).*
available experimental investigations. But more substantial improvements can be only expected from a more sophisticated boundary layer model allowing first the Schmidt number to vary through the turbulent outer region as well. This generalization would increase the mathematical complexity of the problem considerably but must be considered as the last conclusive attempt to test the reliability of the Couette flow type model to predict quantities for the two-dimensional boundary layer.

Ness evaluated also the effect of helium injection on the local energy transfer at the wall. Unfortunately Stanton numbers cannot be evaluated in a relatively simple way from these results so that the prediction of his theory for the heat transfer reduction due to helium injection cannot be compared here with available experimental data.

Another theoretical analysis of the turbulent boundary layer with foreign gas injection has been proposed by Rubesin and Pappas(68). They introduced a theory for the turbulent boundary layer with light gas injection based again on the mixing length concept. Their theory applies only to incompressible, nearly isothermal, turbulent boundary layers so that it can be assumed with reasonable accuracy that the energy transfer at the wall is based on thermal conduction only. The theory is based on the derivation of a modified Reynolds analogy between momentum, heat and mass transfer for a binary system consisting of the main stream fluid air and an injected light gas like helium or hydrogen. They assumed that the turbulent Schmidt and Prandtl number are equal to unity in the turbulent region. In the laminar sublayer the laminar Schmidt number $Sc = \mu/\rho D$ and the laminar Prandtl number $Pr = \mu c_p/\lambda$ were assumed to be constant and were not allowed to vary as in Ness’s theory.

The prediction of Rubesin and Pappas’s theory for the reduction of the skin friction coefficient and the Stanton number due to helium and hydrogen injection is shown for $Re = 10^6$ in Figs. 50 and 51, respectively. The theory predicts a smaller reduction of the skin friction coefficient than found by experiments in incompressible flow. The agreement between theory and experimental data improves slightly for higher Reynolds numbers in the order of $Re = 10^8$. As demonstrated in Fig. 50 hydrogen injection is even more effective in reducing the skin friction coefficient than helium injection. Because of the danger of obtaining highly explosive concentrations of oxyhydrogen gas close to the wall the possibility of injecting hydrogen into the boundary layer is only of academic interest and hence not investigated experimentally, at least to the author’s knowledge.

The prediction of Rubesin and Pappas’s theory for the Stanton number in incompressible flow is not very encouraging as shown in Fig. 51 in
comparison with experimental data. The theory predicts for small injection rates a slight increase of the Stanton number due to helium injection before it decreases for higher injection rates. Experimental data by Pappas and

Okuno show a slight increase of the Stanton number (see Fig. 43) as well. Unfortunately their data are not very conclusive because their values for the Stanton number are subject to a considerable scatter so that only much more thorough experimental investigations can clear the situation. For higher injection rates the theory predicts again a much smaller reduction of the Stanton number due to helium injection than found by experiments either in incompressible flow or in compressible flow at higher Mach

![Graph](image-url)
numbers. The well-defined Reynolds number effect in Figs. 50 and 51 on the reduction of the skin friction coefficient and the Stanton number should be noticed. Such a Reynolds number effect in the same order of magni-

![Graph showing the reduction of the Stanton number due to helium injection.](image)

**Fig. 51.** The reduction of the Stanton number due to helium injection (comparison of Rubesin and Pappas's (66) theory with experimental data).

...tude was not found for air injection neither by Rubesin's theory nor by experimental investigations.

Rubesin and Pappas's theory predicts no effect of helium injection on the recovery factor which is a contradiction to available experimental data which show a decrease of the recovery factor with increasing injection rate very similar to the effect of air injection.

In general it must be said that Rubesin and Pappas's theory is not, by far, as reliable as that of Ness in predicting the skin friction coefficient...
and the Stanton number. Ness's slightly more complex boundary layer model compared with that of Rubesin and Pappas in allowing the Schmidt number to vary through the sublayer and assuming it constant throughout the turbulent outer region having the value at the edge of the sublayer is certainly superior over that of Rubesin and Pappas with a constant Schmidt

![Graph showing the reduction of the Stanton number due to helium injection](image)

**Fig. 52.** The reduction of the Stanton number due to helium injection (comparison of Culick's theory with experimental data).

and Prandtl number in the laminar sublayer and a Schmidt and Prandtl number of unity in the turbulent region.

The theory of Culick lies between those of Ness and Rubesin and Pappas. Culick extended practically the theory of Rubesin and Pappas to compressible flow. Skin friction and heat transfer were again related to each other by a Reynolds analogy. The restriction that the diffusion effect
of helium injection only influences the sublayer is modified, compared with Rubesin and Pappas's theory, to such a degree that the fluid properties in the sublayer (expressed by Prandtl and Schmidt numbers) were assumed to vary with the concentration, while the turbulent Prandtl and Schmidt numbers were assumed unity again. The result of Culick's theory for the

![Graph](image)

**Fig. 53.** The reduction of the skin friction coefficient due to Freon-12 injection (comparison of Pappas's\(^{124}\) theory with experimental data).

Stanton number is presented in Fig. 52 for a Mach number of 3. The agreement with experimental data is quite good but becomes worse with increasing injection rate and is certainly better than for Rubesin and Pappas's theory.

Pappas\(^{124}\) extended Rubesin and Pappas's theory to the case of heavy gas injection which is represented by Freon-12. It is only applicable to
incompressible flow. The result for the skin friction is shown in Fig. 53. Pappas's prediction for the skin friction coefficient lies within the scatter of Pappas and Okuno's experimental data for nearly incompressible flow ($M = 0.3$) but gives a bigger reduction of $c_f$ due to Freon-12 injection than found by the experimental investigation of Romanenko and Kharchenko (65) in incompressible flow.

It is obvious that further theoretical approaches must pay much more attention to the diffusion effects all through the boundary layer, as measurements by Scott et al. (40) for example, have shown. A good starting point is the investigations by Eckert and Schneider (126), Tewfick et al. (127) and Tewfick (128). One-dimensional heat and mass transfer are the experimental and theoretical subjects of these papers. Based on these papers the assumptions of all the theories mentioned above concerning the concentration variation through the boundary layer can be improved accordingly. The necessary information about the fluid properties of gas mixtures (like viscosity, thermal conductivity, Prandtl number, recovery factor) can be found in refs. 129 to 133.

A number of theories deal more directly with film and ablation cooling and will be only mentioned here for comprehensiveness. They are all mainly based again on the mixing length theory. The most important papers are by Lapin (67) for the injection of an inert or reactive coolant into air and by Denison (68) for the system carbon into air. Denison derived his theory for both Prandtl's mixing length concept and Karman's similarity hypothesis without expressing a clear preference.

3.4. Analysis based on similarity parameters

Spalding et al. (60) extended their calculation method for turbulent boundary layers with air injection† to include also the case of foreign gas injection. Their extended method concentrates only on the effect of helium and hydrogen injection but could be in principle modified to predict also the effect of other gases on the boundary layer parameters $c_f$ and $c_h$.

The boundary layer model is the same as for air injection, hence Spalding et al. assume that eq. (2.15)

$$\frac{c_f}{2} \cdot F_c = \psi_o (R_o \cdot F_{Re})$$

† Spalding et al.'s theory for turbulent boundary layers with air injection was discussed in Section 2.1.4.
with
\[
F_c = F_c(M_\infty, T_w/T_\infty, B)
\]
\[
F_{R\theta} = F_{R\theta}(M_\infty, T_w/T_\infty, B)
\]

hold also for the case of foreign gas injection whereby the functions \(F_c\) and \(F_{R\theta}\) have to be determined for every injected gas. It was shown in Section 2.1.4 that the function \(F_c\) can be expressed with reasonable accuracy by
\[
F_c = \left[ \int_0^1 \frac{\sqrt{\theta/\theta_\infty}}{1 + \frac{B u}{u_\infty}} \, d \left( \frac{u}{u_\infty} \right) \right]^{-2}. \quad (2.21)
\]

The main problem of an integration of eq. (2.21) is the determination of the density ratio \(\theta/\theta_\infty\). A relatively simple expression can be obtained for the case of air injection assuming a Prandtl number of unity throughout the boundary layer, namely
\[
\frac{\theta}{\theta_\infty} = \left[ \frac{T_w}{T_\infty} + \left( 1 + \frac{\gamma^2}{2} M_\infty^2 - \frac{T_w}{T_\infty} \right) \frac{u}{u_\infty} - \frac{\gamma^2}{2} M_\infty^2 \left( \frac{u}{u_\infty} \right)^2 \right]^{-1}. \quad (3.28)
\]

In the case of foreign gas injection the diffusion effects must be taken into consideration which was done by Spalding et al. in the most simple way assuming Prandtl and Schmidt numbers of unity throughout the whole boundary layer. This restriction leads to the following expression for the density ratio
\[
\frac{\theta}{\theta_\infty} = \frac{T_w}{T_\infty} \left\{ 1 + \frac{B}{1 + B} \left( \frac{C_{PF}}{C_{P\infty}} - 1 \right) \right\}^{-1}
\]
\[
+ \left\{ 1 + \frac{\gamma^2}{2} M_\infty^2 \cdot \frac{T_w}{T_\infty} \left[ 1 + \frac{B}{1 + B} \left( \frac{C_{PF}}{C_{P\infty}} - 1 \right) \right] \right\} \frac{u}{u_\infty} - \frac{\gamma^2}{2} M_\infty^2 \left( \frac{u}{u_\infty} \right)^2. \quad (3.29)
\]

With the help of eq. (3.29) Spalding et al. integrated eq. (2.21) covering again a big range for their variables, namely
\[
0 \ll M_\infty \ll 12
\]
\[
0.05 \ll T_w/T_\infty \ll 20
\]
\[
0 \ll B \ll 19.
\]
It must be noted again that real gas effects and radiation effects are not considered in their analysis so that the results for hypersonic flow should be used with care. The variation of the parameter $F_c$ is available in relation to the quantity $(1 + \beta)$ in the form of graphs and tables for the injection gases helium and hydrogen whereby two cases are distinguished in the latter case. The possibility of forming oxyhydrogen gas is considered from two extreme points of view:

(i) it is supposed that the pressure and temperature levels are so low that no appreciable amount of combustion occurs;
(ii) an equilibrium state exists throughout the boundary layer assuming the chemical reaction to be sufficiently great.

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Fig. 54. The reduction of the skin friction coefficient due to helium injection (comparison of Spalding et al.'s ([8]) theory with experimental data).
As for the case of air injection it was assumed that the function \(c_f F_C\) is unique and independent of Mach number, temperature ratio and injection gas. It was pointed out in Section 2.1.4 that this approach is rather suspect so that the mean curve in Fig. 13 put through the experimental data points must be regarded as a very rough approximation. All the further calculation in determining the skin friction coefficient and the Stanton number is identical to the method described already for the case of air injection.

The prediction of the theory for the reduction of the skin friction coefficient due to helium injection is presented in Fig. 54. The agreement with experimental data is quite good for incompressible flow. Unfortunately the theory predicts an increasing reduction of the skin friction ratio \(c_f/c_{f_0}\) for constant \(2F/c_{f_0}\) with increasing Mach number which was not found by any experimental investigation nor by any other theory. These wrong predictions must be attributed chiefly to the simplifications introduced by the relationship for \(F_{Re}\).

Similarly bad are the predictions of the theory for the Stanton number as shown in Fig. 55. Again the predicted tendency is wrong compared with experimental data in that the ratio \(c_h/c_{h_0}\) decreases with increasing Mach number for constant \(F/c_{h_0}\). For the case of incompressible flow the theory gives a slightly bigger reduction of the Stanton number ratio than found by experiment. With increasing Mach number the theory becomes more and more unreliable.

But nevertheless, Spalding et al.'s theory might be interesting for quick engineering calculations mainly because of its simplicity which makes it quite useful for rough estimations of the skin friction coefficient and the Stanton number. The mathematical simplicity is its main advantage over the other available theories so that it might be worth improving Spalding et al.'s calculation concept by recalculating the function \(F_c\) and \(F_{Re}\) based on an improved boundary layer model with varying Schmidt and Prandtl numbers for example. Moreover, the effect of Mach number, temperature ratio and foreign gas injection on the function \(F_{Re}\) should be reconsidered and revised.

3.5. Final remarks

Apart from Ness's theory the agreement between the various other theories and the experimental investigations is not very convincing especially for higher injection rates. But it must be noted here that the experimental data are subject to considerable errors and are mainly measured at cones rather than flat plates, the geometrical configuration used by all theories.
Despite the scatter of the theories as well as the experimental results, it can be deduced that light gas injection (like helium or hydrogen) results in a bigger reduction of the skin friction coefficient and the Stanton number when compared with air injection. The injection of heavy gases like Freon-12 also results in a reduction of the skin friction coefficient and the Stanton number, but the effect is smaller than for air injection.

4. SUGGESTIONS FOR FUTURE RESEARCH

Considering the success so far obtained with Coles's and Jeromin's transformation it would be interesting to investigate the possibilities of boundary layer transformations further by extending them to compressible
flows with and without air injection, heat transfer and pressure gradients. The behaviour of the incompressible flow must be known before the transformation can be applied to compressible flow (see, for example, Jeromin\(^{(8)}\)). Consequently a topic of further research must be the detailed study of the corresponding incompressible flows with air injection taking into account the additional effects of heat transfer and pressure gradients on the boundary layer behaviour. Future investigations should not only concentrate on the usual boundary layer parameters like skin friction and heat transfer coefficient but should measure velocity and temperature profiles as well. In general, experimental data should be made available in the form of tables rather than only graphs so that they can be used for further detailed analyses which were not the subject of the original investigation.

More attention should be paid to the development of similarity laws for the temperature variation through the boundary layer similar to the law of the wall and the velocity defect law for the velocity distribution. A very interesting topic in incompressible flow might be a general detailed investigation of the way in which the concepts of the law of the wall and the velocity defect law can be modified to consider the effects of heat transfer, pressure gradients and air injection so that these results could be applied to Jeromin’s transformation, for example and so their range of application extended to compressible flow as well. Another way to consider the influence of pressure gradients and air injection on the incompressible turbulent boundary layer development is the use of McQuaid’s\(^{(108)}\) calculation method which is valid for zero heat transfer. This approach would lead to the calculation of the velocity profiles of a whole boundary layer development and so give the necessary boundary layer parameters in order to determine the transformation parameters. Torii’s\(^{(107)}\) calculation method includes the case of heat transfer at the wall so that even this influence on the transformation parameter could be investigated calculating the velocity and temperature profiles simultaneously. Both calculation methods are rather tedious and necessitate an enormous amount of computing work before they can be applied. But in principle one should be able to use these theories to study the effects of pressure gradients and heat transfer on the transformation.

More experimental data are also necessary for transpired incompressible boundary layers at high Reynolds numbers since the transformed compressible flows normally lie in a higher Reynolds number range than covered by data for incompressible flow. In compressible flow more detailed boundary layer studies should be carried out to consider the effect of
pressure gradients and heat transfer on the boundary layer development. This is preferable to the measurements of skin friction and Stanton numbers such as carried out by Pappas and Okuno (32) and Fogaroli and Saydah (43). If feasible a higher Reynolds number range should be covered than was possible in Jeromin’s experimental investigation (23).

A very interesting topic would be the detailed study of foreign gas injection to simulate to a certain extent film and ablation cooling. These investigations should be carried out simultaneously in incompressible and compressible flow. A theoretical topic might be an extension of Jeromin’s transformation to foreign gas injection. Similarity laws for the velocity, temperature and viscosity distribution through the incompressible boundary layer in the presence of foreign gas injection must be developed before such an approach becomes attractive since it is the aim of a transformation to extend the use of available formulae for incompressible flow to compressible flow. At present there is no law which describes the incompressible turbulent boundary layer with foreign gas injection so that it would be useful first of all to determine the way in which a concept such as Stevenson’s law of the wall can be adapted to foreign gas injection by adjusting, for example, the constants \( \alpha \) and \( C \) suitably.

5. ACKNOWLEDGEMENTS

The author wishes to express his gratitude to his supervisor Dr. L. C. Squire for his continued interest, assistance and encouragement during the whole course of research. Many useful discussions with him have contributed to this work. Thanks are also due to Dr. J. McQuaid for his kindness in allowing me to abstract the results of his theoretical and experimental investigations before they have been published.

I should like to express my particular thanks to the Provost and Fellows of King’s College, Cambridge, for their award of an external studentship and for their support throughout the three academic years of the research.

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